C. R. Math. Rep. Acad. Sci. Canada Vol. **32** (4) 2010, pp. 120–126

MORSE EQUI-SINGULAR DEFORMATION IN $\mathbb{C}^2$

TZEE-CHAR KUO AND LAURENTIU PAUNESCU

Presented by Pierre Milman, FRSC

ABSTRACT. The *enriched complex line* $\mathbb{C}_*$ is $\mathbb{C}$ plus a set of *infinitesimals*. The Morse stability notion is used for an *equi-singular deformation theorem* in $\mathbb{C}\{x,y\}$ $(=\mathcal{O}_2)$.

RÉSUMÉ. On appelle *la droite complexe enrichie* C_*, la réunion de la droite complexe $\mathbb{C}_*$ avec un ensemble des infinitésimaux. La notion de stabilité de Morse est appliquée au théorème de déformation equisingulière dans $\mathbb{C}\{x,y\}$ $(=\mathcal{O}_2)$.

A principle we believe in is that the study of a given $f(x,y) \in \mathbb{C}\{x,y\}$ can be reduced to that of a polynomial in one variable. This is explained as follows.

First, apply a linear transformation, if necessary, so that f is *mini-regular* in x, *i.e.*,

$$f(x,y) = \sum a_{ij}x^i y^j = H_m(x,y) + H_{m+1}(x,y) + \cdots, \quad H_m(1,0) \neq 0, \tag{0.1}$$

where $m = O(f)$. Then, by the Newton–Puiseux Theorem,

$$f(x,y) = \text{unit} \cdot \prod_{i=1}^{m} \big(x - \alpha_i(y)\big), \quad f_x(x,y) = \text{unit} \cdot \prod_{j=1}^{m-1} \big(x - \gamma_j(y)\big), \tag{0.2}$$

where $\alpha_i(y)$, $\gamma_j(y)$ are (convergent) fractional power series, $O_y(\alpha_i), O_y(\gamma_j) \geq 1$.

Thus, the study of $f(x,y)$ and the "polar" $f_x(x,y)$ are reduced to that of the polynomials on the right hand side. Of course, the coefficients of the polynomials are functions of y (as in the Weierstrass Preparation Theorem).

In this paper, the notion of Morse stability is "transplanted" into algebraic geometry, and, in accordance with the principle, an equi-singular deformation theorem in $\mathbb{C}^2$ is formulated as a kind of the Morse Stability Theorem for polynomials in one variable ξ. (Compare [2].)

To expose the ideas, consider the following two t-parametrized deformations:

$$Q_t(x,y) = x^4 - 2t^2x^2y^d + y^{2d}, \quad P_t(x,y) = x^3 - y^4 - 3t^2xy^{2d}, \tag{0.3}$$

where $d \geq 2$, $|t| < \epsilon$. Both are Milnor μ-constant [4] (topologically equi-singular).

Received by the editors on May 28, 2010; revised July 16, 2010.
AMS Subject Classification: 14HXX, 32SXX, 58K60, 58K40.

When the stability notion is introduced in Definition 1.3 and Definition 2.3, we shall see that $\{Q_t\}$ is not equi-singular in our sense, because $\xi^4-2t^2\xi^2y^d+y^{2d}$, as a deformation of ξ^4+y^{2d}, is not stable: the critical point $\xi=0$ at $t=0$ splits into three, $\xi=0$, $\pm ty^{d/2}$ when $t\neq 0$. The trivialization in the Main Theorem does not exist for Q_t.

The associated deformation of P_t is ξ^3-y^4; being independent of t, it is obviously Morse stable. Thus the Pham family $P_t(x,y)$ [5] admits the trivialization in the Main Theorem, which is much stronger than a mere topological trivialization of the variety $P_t(x,y)=0$.

The details of this paper can be found in [3].

Conventions. Throughout this paper, by "$+\cdots$" we mean "*plus higher order terms*".

We say $\varphi(w)$ is *real analytic*, $w=u+\sqrt{-1}\,v\in\mathbb{C}$, if it is so as a function of $(u,v)\in\mathbb{R}^2$.

Finally, $\epsilon>0$ is a sufficiently small constant,

$$I_{\mathbb{R}}:=\{t\in\mathbb{R}\mid |t|<\epsilon\},\quad I_{\mathbb{C}}:=\{t\in\mathbb{C}\mid |t|<\epsilon\},\quad I_{\mathbb{F}}:=\{t\in\mathbb{D}\mid |t|<\epsilon\}.$$

($\mathbb{D}$ is defined below.)

1. Newton–Puiseux analysis.

The classical Newton–Puiseux Theorem asserts that the field $\mathbb{F}$ of convergent fractional power series in an indeterminate y is algebraically closed [6], [7].

Recall that a non-zero element of $\mathbb{F}$ is a (finite or infinite) convergent series

$$\alpha:\alpha(y)=a_0y^{n_0/N}+\cdots+a_iy^{n_i/N}+\cdots,\quad n_0<n_1<\cdots, \tag{1.1}$$

where $N\in\mathbb{Z}^+$, $n_i\in\mathbb{Z}$, $0\neq a_i\in\mathbb{C}$. We can assume $\mathrm{GCD}(N,n_0,n_1,\dots)=1$.

The *order* of α is $O_y(\alpha):=n_0/N$, $O_y(0):=+\infty$; the *Puiseux multiplicity* is $m_{\mathrm{puis}}(\alpha):=N$. The *conjugates* of α are

$$\alpha^{(k)}_{\mathrm{conj}}(y):=\sum a_i\theta^{kn_i}y^{n_i/N},\quad 0\le k\le N-1,\quad \theta:=e^{2\pi\sqrt{-1}/N}.$$

The following $\mathbb{D}$ is an integral domain with quotient field $\mathbb{F}$ and ideals $\mathbb{M}$, $\mathbb{M}_1$:

$$\mathbb{D}:=\{\alpha\in\mathbb{F}\mid O_y(\alpha)\ge 0\},\quad \mathbb{M}:=\{\alpha\mid O_y(\alpha)>0\},\quad \mathbb{M}_1:=\{\alpha\mid O_y(\alpha)\ge 1\}.$$

Define $|\alpha|:=\sum 2^{-n_i/N}|a_i|(1+|a_i|)^{-1}$. Then $d(\alpha,\beta):=|\alpha-\beta|$ is a *metric* on $\mathbb{D}$. If we fix N, then $\lim_{m\to\infty}\sum a_i(m)y^{n_i/N}=0$ *if and only if* each $a_i(m)\to 0$ (point-wise convergence).

Given $U\subset\mathbb{D}$, open, and $\phi\colon U\to\mathbb{F}$. We say ϕ is *Puiseux–Lojasiewicz bounded* if every $\alpha\in U$ has a neighbourhood $\mathcal{N}(\alpha)$ with constants $K(\alpha),L(\alpha)>0$, such that

$$m_{\mathrm{puis}}\big(\phi(\xi)\big)\le K(\alpha)m_{\mathrm{puis}}(\xi),\quad O_y\big(\phi(\xi)\big)\ge -L(\alpha),\ \xi\in\mathcal{N}(\alpha).$$

For example, $\sum y^{1/n}\xi^n$ and $\sum y^{-n}\xi^n$ are not Puiseux–Lojasiewicz bounded.

In this paper we only study functions which are Puiseux–Lojasiewicz bounded.

Definition 1.1 We say ϕ is *differentiable* at $\alpha \in U$, with *derivative* $\phi'(\alpha) \in \mathbb{F}$, if

$$\phi'(\alpha) = \lim[\phi(\alpha+\delta) - \phi(\alpha)]/\delta \quad \text{as } \delta \to 0 (\delta \in \mathbb{D}).$$

If $\phi'(\gamma) = 0$, γ is a *critical point*, with *multiplicity*

$$m_{\text{crit}}(\gamma) := \max\{k \mid \phi^{(i)}(\gamma) = 0, 1 \le i \le k\}.$$

Cauchy's Theorem. *If $\phi'(\alpha)$ exists at every $\alpha \in U$, then all derivatives $\phi^{(k)}(\alpha)$ exist,*

$$\text{(1.2)} \qquad \oint_{z\in C} \phi(\mu)\, d\mu = 0, \quad \phi^{(k)}(\alpha) = \frac{k!}{2\pi\sqrt{-1}} \oint_{z\in C} \frac{\phi(\mu)}{(\mu-\alpha)^{k+1}}\, d\mu, \quad k \ge 0,$$

where $\mu := \alpha + z\delta$, $\delta \in \mathbb{D}$, $d\mu := \delta dz$, C a sufficiently small contour around $0 \in \mathbb{C}$.

Moreover, ϕ is $\mathbb{F}$-analytic *in the sense that if $\alpha + z\delta \in U$, $|z| < r$, $z \in \mathbb{C}$, then*

$$\text{(1.3)} \qquad \phi(\alpha + z\delta) = \phi(\alpha) + \cdots + (1/k!)\phi^{(k)}(\alpha)(z\delta)^k + \cdots, \quad |z| < r,$$

where $m_{\text{puis}}\big(\phi^{(k)}(\alpha)\big) \le K(\alpha)$, a constant.

Consider a given $\phi\colon \mathbb{M}_1 \to \mathbb{M}_1$, which *extends* to a differentiable function $U \to \mathbb{D}$, U open. Taking $\alpha, \delta \in \mathbb{M}_1$, $\xi := z\delta$, we have the "Taylor expansion" of ϕ at α:

$$\text{(1.4)} \qquad \phi(\alpha + \xi) = \sum \alpha_k \xi^k, \quad \xi \in \mathbb{M}_1, \quad \alpha_k := (1/k!)\phi^{(k)}(\alpha) \in \mathbb{D}.$$

In this paper we always assume ϕ is *mini-regular in* ξ, say of *order* m, *i.e.*,

$$\text{(1.5)} \qquad O_y(\alpha_m) = 0, \quad O_y(\alpha_k) + k \ge m \quad \text{for } 0 \le k \le m-1.$$

By the Newton–Puiseux Theorem, ϕ has m roots in $\mathbb{M}_1$ (counting multiplicities),

$$\text{(1.6)} \qquad Z(\phi) := \{\zeta \in \mathbb{M}_1 \mid \phi(\zeta) = 0\} := \{\zeta_1, \ldots, \zeta_d\}, \quad m = \sum m_i,$$

where $m_i := m(\zeta_i)$ is the multiplicity of ζ_i. Of course ϕ has $m-1$ critical points in $\mathbb{M}_1$.

In $\mathbb{M}_1$, define $\mu \sim_\phi \nu$ *if and only if* either $\mu = \nu$ or else

$$O_y(\mu - \nu) > O_y(\mu - \zeta_i) = O_y(\nu - \zeta_i), \quad 1 \le i \le d.$$

The equivalence class of μ is denoted by μ_ϕ, with *height* and *Lojasiewicz exponent*

$$h(\mu_\phi) := \max\{O_y(\mu - \zeta_i) \mid \zeta_i \in Z(\phi)\}, \quad L(\mu_\phi) := \sum O_y(\mu - \zeta_i),$$

respectively. The quotient space is $\mathbb{M}_{1,\phi} := \mathbb{M}_1/{\sim_\phi}$.

Let $\mu_\phi(y)$ denote $\mu(y)$ with terms y^e deleted, $e > h(\mu_\phi)$. We call $\mu_\phi(y) \in \mathbb{M}_1$ the *canonical coordinate* of $\mu_\phi \in \mathbb{M}_{1,\phi}$; μ_ϕ and $\mu_\phi(y)$ are often *identified*; $O(\mu_\phi - \nu_\phi)$ is well-defined.

For example, if $\phi(\xi) := \xi^2 - 2y^3$, $\mu(y) := y^{3/2} + y^{7/4}$, then $\mu_\phi(y) = y^{3/2}$, $h(\mu_\phi) = 3/2$.

Let ϕ be as above. By an $\mathbb{F}$-analytic *deformation* of ϕ we mean

$$\Phi(\xi, t) := \phi_t(\xi) := \sum A_i(t)\xi^i \in \mathbb{F}\{\xi, t\}, \quad \Phi(\xi, 0) = \phi(\xi), \tag{1.7}$$

where $A_i(t) \in \mathbb{D}\{t\}$, $t \in I_{\mathbb{F}}$, and Φ is mini-regular: $O_y\big(A_i(t)\big)+i \geq m$, $0 \leq i \leq m$.

In $\mathbb{M}_1 \times I_{\mathbb{F}}$, define $(\mu, t) \sim_\Phi (\nu, t')$ *if and only if* $t = t'$, $\mu_{\phi_t} = \nu_{\phi_t}$. The quotient space is $\mathbb{M}_1 \times_\Phi I_{\mathbb{F}} := \mathbb{M}_1 \times I_{\mathbb{F}}/{\sim_\Phi}$.

If γ is a critical point of ϕ, we call $\gamma_\phi \in \mathbb{M}_{1,\phi}$ a *blurred critical point*. The multiplicity $m_{\text{crit}}(\gamma_\phi)$ is, by definition, the total number of $\mu \in \mathbb{M}_1$ such that $\phi'(\mu) = 0$, $\mu_\phi = \gamma_\phi$.

Example 1.2 For $f = (x^2 - y^3)^2 - 8xy^5$, ϕ has three critical points: $\gamma^\pm = \pm y^{3/2} + y^2 + \cdots$, and $\gamma_0 = -2y^2 + \cdots$, where $m_{\text{crit}}(\gamma_\phi^+) = m_{\text{crit}}(\gamma_\phi^-) = m_{\text{crit}}(\gamma_0) = 1$.

Definition 1.3 We say Φ is *almost Morse stable* if the following holds: Each blurred critical point γ_ϕ admits a continuous deformation

$$\tau \colon I_{\mathbb{F}} \to \mathbb{M}_1 \times_\Phi I_{\mathbb{F}}, \quad t \mapsto \big(\tau_t(\gamma_\phi), t\big), \quad \tau_0(\gamma_\phi) = \gamma_\phi, \tag{1.8}$$

such that $\tau_t(\gamma_\phi) \in \mathbb{M}_{1,\phi_t}$ is a blurred critical point (of ϕ_t), and

$$m_{\text{crit}}\big(\tau_t(\gamma_\phi)\big) = m_{\text{crit}}(\gamma_\phi), \quad L\big(\tau_t(\gamma_\phi)\big) = L(\gamma_\phi). \tag{1.9}$$

(We can show that (1.9) implies $h\big(\tau_t(\gamma_\phi)\big) = h(\gamma_\phi)$.)

Example 1.4 For $P_t(x, y)$ in (0.1), $\Phi(\xi, t) = \xi^3 - y^4 - 3t^2y^{2d}\xi$ and $\gamma_{\phi_t} = 0$ is the *unique* blurred critical point, $m_{\text{crit}}(\gamma_{\phi_t}) = 2$. With $\tau_t = \text{id}$, Φ is stable.

Attention: ϕ_t has two critical points $\pm ty^d$ when $t \neq 0$; but γ_{ϕ_t} is unique.

Another example: $\Psi(\xi, t) := \xi^3 + ty^3 - y^4$ is *not almost Morse stable*, (1.9) is not satisfied.

2. **Main theorem.**

A *real analytic* map-germ $\rho \colon \big([0, \infty), 0\big) \to (\mathbb{C}^2, 0)$ is called an *analytic arc* [4]; we call the *image set-germ*, $\text{Im}(\rho)$, a *geo-arc*. The *complexification* of ρ is $\rho_{\mathbb{C}}(z) := \rho(z)$, $z \in \mathbb{C}$.

Given $k \in \mathbb{Z}^+$, define $\rho^{(k)}(s) := \rho(s^k)$. Then $\text{Im}(\rho^{(k)}) = \text{Im}(\rho)$, the same geo-arc.

Take $f(x,y) \in \mathbb{C}\{x,y\}$ as in (0.1), $\phi(\xi) := f(\xi, y)$. Take $F_t(x,y) := F(x,y,t) \in \mathbb{C}\{x,y,t\}$ such that $\Phi(\xi,t) := F_t(\xi,y)$ is a deformation of $\phi(\xi)$ in the sense of (1.7).

Main Theorem, Part I. *Suppose the deformation $\Phi(\xi,t)$ is almost Morse stable. Then there exists a t-level preserving homeomorphism*

$$(2.1)\qquad H\colon (\mathbb{C}^2 \times I_{\mathbb{C}}, 0 \times I_{\mathbb{C}}) \to (\mathbb{C}^2 \times I_{\mathbb{C}}, 0 \times I_{\mathbb{C}}), \quad \big((x,y),t\big) \mapsto \big(H_t(x,y),t\big),$$

which is real bi-analytic outside $\{0\} \times I_{\mathbb{C}}$. The following hold.
(1) *$F\big(H_t(x,y),t\big) = f(x,y)$, $t \in I_{\mathbb{C}}$,* i.e., *$F(x,y,t)$ is "trivialized" by H.*
(2) *There exists $c > 0$, $c \le \|H_t(x,y)\|/\|(x,y)\| \le 1/c$, $t \in I_{\mathbb{C}}$.*
(3) *If $\rho(s)$ is an analytic arc, then for some $k \in \mathbb{Z}^+$, $H_t\big(\rho^{(k)}(s)\big)$ is a "geo-arc wing" in the sense that $H_t\big(\rho^{(k)}(s)\big)$ is real analytic in (s,t). In particular, the flow H_t carries geo-arcs to geo-arcs.*

Now the geometric implications. Take a holomorphic map-germ $\mathcal{A}\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$, $\mathcal{A}(z) \ne 0$ if $z \ne 0$. The *image set-germ*, $\mathrm{Im}(\mathcal{A})$, or *the geometric locus* of $\mathcal{A}$, has a well-defined tangent line $T(\mathcal{A})$ at 0. We call $\mathrm{Im}(\mathcal{A})$ an *infinitesimal at $T(\mathcal{A}) \in \mathbb{C}P^1$*.

For example, the curve-germs $x^2 - y^3 = 0$, $x^2 - y^5 = 0$ are infinitesimals at $[0:1]$.

The set of infinitesimals is $\mathbb{C}P^1_*$, and, as in Projective Geometry, $\mathbb{C}P^1_* = \mathbb{C}_* \cup \mathbb{C}'_*$,

$$\mathbb{C}_* := \{\mathrm{Im}(\mathcal{A}) \mid T(\mathcal{A}) \ne [1:0]\}, \quad \mathbb{C}'_* := \{\mathrm{Im}(\mathcal{A}) \mid T(\mathcal{A}) \ne [0:1]\}.$$

The geometric locus of $z \mapsto (az, bz)$ is identified with $[a:b] \in \mathbb{C}P^1$, hence $\mathbb{C}P^1 \subset \mathbb{C}P^1_*$.

Given $\alpha \in \mathbb{M}_1$, let $\mathcal{A}(z) := \big(\alpha(z^N), z^N\big)$, $\alpha_* := \pi_*(\alpha) := \mathrm{Im}(\mathcal{A})$. We use $\pi_*\colon \mathbb{M}_1 \to \mathbb{C}_*$ as a *coordinate system* on $\mathbb{C}_*$, and furnish $\mathbb{C}_*$ with the quotient topology.

The *contact order* is: $\mathcal{C}_{\mathrm{ord}}(\alpha_*, \alpha_*) := \infty$, and if $\alpha_* \ne \beta_*$, then

$$(2.2)\qquad \mathcal{C}_{\mathrm{ord}}(\alpha_*, \beta_*) := \max_j\{O_y(\alpha - \beta^{(j)}_{\mathrm{conj}})\} = \max_{k,j}\{O_y(\alpha^{(k)}_{\mathrm{conj}} - \beta^{(j)}_{\mathrm{conj}})\}.$$

This is also the smallest number L [1] such that

$$d(x,y) \ge a\|(x,y)\|^L, \quad x \in \alpha_*,\ y \in \beta_*,\ \|x\| = \|y\|,\ a > 0.$$

The *Puiseux (characteristic) pairs* of α_* [7], which describes the iterated torus knot of the curve-germ α_*, is denoted by $\chi_{\mathrm{puis}}(\alpha)$ or $\chi(\alpha_*)$.

The *enriched complex plane* is $\mathbb{C}_*$ with the above structures. (The Riemann–Zariski surface is much larger than $\mathbb{C}_*$, *e.g,*, $x = y^{\sqrt{2}}$ defines a point in the former, but not in the latter.)

In $\mathbb{C}_*$, define $\alpha_* \sim_f \beta_*$ *if and only if* either $\alpha_* = \beta_*$, or else

$$\mathcal{C}_{\text{ord}}(\alpha_*, \beta_*) > \mathcal{C}_{\text{ord}}(\alpha_*, \zeta_*) = \mathcal{C}_{\text{ord}}(\beta_*, \zeta_*) \quad \forall \zeta \in Z(\phi).$$

The equivalence class of α_*, denoted by $\alpha_{*/f}$, is called a *blurred* point. (If $\zeta \in Z(\phi)$ then $\zeta_{*/f} := \{\zeta_*\}$, a singleton.) The quotient space is $\mathbb{C}_{*/f} := \mathbb{C}_*/\sim_f$.

We call $\alpha_\phi(y)$, or any conjugate $\alpha^{(k)}_{\phi,\text{conj}}(y)$, a *canonical coordinate* of $\alpha_{*/f}$.

If $\phi'(\gamma) = 0$, then $\gamma_{*/f}$ is called a *blurred* critical point and the *multiplicity*, $m_{\text{crit}}(\gamma_{*/f})$, is the total number of $\mu \in \mathbb{M}_1$, counting multiplicities, such that $\phi'(\mu) = 0$, $\mu_{*/f} = \gamma_{*/f}$.

Define height $h(\alpha_{*/f}) := h(\alpha_\phi)$, Puiseux pairs $\chi_{\text{puis}}(\alpha_{*/f}) := \chi_{\text{puis}}(\alpha_\phi)$, and contact order

$$\mathcal{C}_{\text{ord}}(\alpha_{*/f}, \beta_{*/f}) := \mathcal{C}_{\text{ord}}(\alpha_*, \beta_*) \quad \text{if } \alpha_{*/f} \neq \beta_{*/f};\ \mathcal{C}_{\text{ord}}(\alpha_{*/f}, \alpha_{*/f}) := \infty.$$

Example 2.1 Let $f(x,y) := x^2 - 2y^3$. Then $x^2 - y^3 = 0$ and $(x^2 - y^3)^2 - 2xy^5 = 0$ define the same blurred point $\alpha_{*/f}$ in $\mathbb{C}_{*/f}$, with $\alpha_\phi = \pm y^{3/2}$, $\chi(\alpha_{*/f}) = \{\frac{3}{2}\}$.

Note that $\chi(y^{3/2} + y^{7/4} + \cdots) = \{\frac{3}{2}, \frac{7}{4}\}$. That is, in general, $\chi(\alpha) \neq \chi(\alpha_\phi)$.

For the function f in Example 1.2, $\gamma^+_{*/f} = \gamma^-_{*/f}$, $m_{\text{crit}}(\gamma^+_{*/f}) = 2$.

Main Theorem, Part II. *In addition, we have:*

(4) *Take $\alpha_{*/f} \in \mathbb{C}_{*/f}$, and any ρ such that* $\text{Im}(\rho_{\mathbb{C}}) \in \alpha_{*/f}$. *Let $\delta_* \in \mathbb{C}_*$ denote the unique curve-germ containing the geo-arc* $\text{Im}(H_t \circ \rho)$. *Then $\eta_t(\alpha_{*/f}) := \delta_{*/F_t}$ is independent of the choice of ρ, and hence $\eta_t \colon \mathbb{C}_{*/f} \to \mathbb{C}_{*/F_t}$ is well-defined. The mapping*

$$\eta_* \colon \mathbb{C}_{*/f} \times I_{\mathbb{C}} \to \mathbb{C}_* \times_F I_{\mathbb{C}}, \quad (\alpha_{*/f}, t) \mapsto \big(\eta_t(\alpha_{*/f}), t\big), \tag{2.3}$$

is a homeomorphism, preserving height, contact order, and Puiseux pairs.

(5) *If $\gamma_{*/f}$ is a blurred critical point in $\mathbb{C}_{*/f}$, then $\eta_t(\gamma_{*/f})$ is one in $\mathbb{C}_{*/F_t}$,*

$$m_{\text{crit}}(\gamma_{*/f}) = m_{\text{crit}}\big(\eta_t(\gamma_{*/f})\big).$$

Take $\phi = \text{id} \colon \xi \mapsto \xi$. We call $\mathcal{V} := \mathbb{M}_{1,\text{id}}$ the *value space*, and $\mathcal{V}_* := \mathbb{C}_{*/\,\text{id}}$ the *geometric value space*. If $\mu(y) = uy^h + \cdots$, $u \neq 0$, μ_{id} is *completely determined* by the pair (u,h). We can identify μ_{id} with (u,h). We also write $0_{\text{id}} := 0_\nu := 0y^\infty$, the “zero” element of $\mathcal{V}$.

The *valuation function* on $\mathbb{M}_{1,\phi}$ is, by definition,

$$\text{val} \colon \mathbb{M}_{1,\phi} \longrightarrow \mathcal{V}, \quad \text{val}(\xi_\phi) := \phi(\xi)_{\text{id}}.$$

Example 2.2 If $\phi(\xi) = \xi^4(\xi - y)^5$, $\varepsilon \neq 0$, $h > 1$, then $\text{val}(\varepsilon y^h + \cdots) = (-\varepsilon^4, 4h+5)$, $\text{val}\big((1+\varepsilon)y + \cdots\big) = \big(\varepsilon^5(1+\varepsilon)^4, 9\big)$, $\text{val}(0) = \text{val}(y) = 0_{\mathcal{V}}$.

The coordinate map $\pi_*\colon \mathbb{M}_1 \to \mathbb{C}_*$ induces π_ϕ and $\pi_{\mathcal{V}}$ in the diagramme

$$\begin{array}{ccc} \mathbb{M}_{1,\phi} & \xrightarrow{\text{val}} & \mathcal{V} \\ {\scriptstyle \pi_\phi}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{V}}} \\ \mathbb{C}_{*/f} & \xrightarrow{\text{val}_*} & \mathcal{V}_* \end{array}$$

where val_* is the *unique* mapping such that the diagramme is commutative.

Definition 2.3 We say an almost Morse stable deformation Φ (see Definition 1.3) is *Morse stable* if the following also holds. Let γ_ϕ, γ'_ϕ be blurred critical points such that

$$\text{val}(\gamma_\phi) = \text{val}(\gamma'_\phi), \quad O(\gamma_\phi - \gamma'_\phi) \geq h(\gamma_\phi) = h(\gamma'_\phi).$$

Then these properties are preserved along the deformation:

$$\text{val}(\gamma_{\phi_t}) = \text{val}(\gamma'_{\phi_t}), \quad O(\gamma_{\phi_t} - \gamma'_{\phi_t}) \geq h(\gamma_{\phi_t}) = h(\gamma'_{\phi_t}).$$

Main Theorem, Part III. *Suppose Φ is Morse stable. Then the following holds.*

(6) *Let $\gamma_{*/f}$, $\gamma'_{*/f}$ be blurred critical points (in $\mathbb{C}_{*/f}$) such that*

$$\text{val}_*(\gamma_{*/f}) = \text{val}_*(\gamma'_{*/f}) \quad \textit{and} \quad \mathcal{C}_{\text{ord}}(\gamma_{*/f}, \gamma'_{*/f}) = h(\gamma_{*/f}) = h(\gamma'_{*/f}).$$

Then these equalities are preserved by η_t, i.e., $\text{val}_*\big(\eta_t(\gamma_{*/f})\big) = \text{val}_*\big(\eta_t(\gamma'_{*/f})\big)$ *and*

$$\mathcal{C}_{\text{ord}}\big(\eta_t(\gamma_{*/f}), \eta_t(\gamma'_{*/f})\big) = h\big(\eta_t(\gamma_{*/f})\big) = h\big(\eta_t(\gamma'_{*/f})\big).$$

References

1. E. Bierstone and P. Milman, *Semianalytic and subanalytic sets.* Inst. Hautes Etudes Sci. Publ. Math. **67** (1988), 5–42.
2. T.-C. Kuo and L. Paunescu, *Equisingularity in R^2 as Morse stability in infinitesimal calculus.* Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), 115–120.
3. ———, *Enriched Riemann Sphere, Morse Stability and Equi-singularity in O_2.* http://arxiv.org/abs/0907.0105.
4. J. Milnor, *Singular Points of Complex Hypersurfaces.* Annals of Mathematical Studies **61**, Princeton Univ. Press, Princeton, 1968.
5. F. Pham, *Deformations equisingularitiés des ideaux jacobiens de courbes planes.* In: Proc. Liverpool Singularities Sym. II (ed. C. T. C. Wall), Lecture Notes in Math. **209** (1971), 218–233, Springer-Verlag.
6. R. J. Walker, *Algebraic Curves.* Springer-Verlag, 1972.
7. C. T. C. Wall, *Singular Points of Plane Curves.* London Mathematical Society Student Texts **63**. Cambridge University Press, Cambridge, 2004.

School of Mathematics
University of Sydney
Sydney, NSW, 2006
Australia
email: tck@maths.usyd.edu.au
email: laurent@maths.usyd.edu.au