EXACT SEQUENCES FOR
EQUIVARIANTLY FORMAL SPACES

MATTHIAS FRANZ AND VOLKER PUPPE

Presented by Eckhard Meinrenken, FRSC

ABSTRACT. Let $T$ be a torus. We present an exact sequence relating the relative equivariant cohomologies of the skeletons of an equivariantly formal $T$-space. This sequence, which goes back to Atiyah and Bredon, generalizes the so-called Chang–Skjelbred lemma. As coefficients, we allow prime fields and subrings of the rationals, including the integers. We extend to the same coefficients a generalization of this “Atiyah–Bredon sequence” for actions without fixed points which has recently been obtained by Goertsches and Töben.


1. Introduction. Let $T = (S^1)^n$ be a torus of dimension $n$ and $X$ a “nice” $T$-space. Suppose that $X$ is cohomologically equivariantly formal\(^1\)(or CEF), which means that its equivariant cohomology $H^*_T(X)$ (with rational coefficients) is free over the polynomial ring $H^*(BT)$. Then the so-called Chang–Skjelbred lemma [CS, (2.3)] asserts that the sequence

$$0 \to H^*_T(X) \to H^*_T(X^T) \to H^*_{T,+1}(X_1, X^T)$$

is exact, where $X^T \subset X$ denotes the fixed point set and $X_1$ the union of all orbits of dimension at most 1. In other words, $H^*_T(X)$ coincides, as subalgebra of $H^*_T(X^T)$, with the image of $H^*_T(X_1) \to H^*_T(X^T)$. Usually $X^T$ and $X_1$ are much simpler than $X$, so that (1.1) gives an easy method to compute $H^*_T(X)$.

\(^1\)It seems that the terminology ‘equivariantly formal’ was coined by [GKM]. We add ‘cohomologically’ to avoid confusion with notions in rational homotopy theory.

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the cohomology of smooth projective toric varieties, because they are known to
be equivariantly formal, see Remark 2.3. There are many other applications of
the Chang–Skjelbred lemma in algebraic and symplectic geometry, often with
implications for combinatorics.

Roughly at the same time as Chang and Skjelbred, Atiyah proved a much
more general theorem in the context of equivariant $K$-theory [A, Chapter 7].
Bredon [Br] then observed that it applies equally to cohomology. In the con-
text of toric varieties an Atiyah-like theorem was proven by Barthel–Brasselet–
Fieseler–Kaup for cohomology and intersection homology [BBFK, Theorem 4.3].
Recently Goertsches and Töben [GT] made the very interesting observation that
the Cohen–Macaulay property, which Atiyah used in an essential way, gives gen-
eralizations also for fixed-point-free actions.

The principal aim of this note is to present a generalized cohomological version
of Atiyah’s theorem. Our proof in Section 2 will be a modification of his, adapted
to cohomology and with a special emphasis on coefficient rings $R$ other than the
rationals, namely on subrings of $\mathbb{Q}$ and on prime fields. As a corollary, we get
an integral version of the Chang–Skjelbred lemma with certain restrictions on
the isotropy groups. We also extend the Goertsches–Töben result to the same
coefficient rings and to more general spaces than compact $T$-manifolds.

The present note is a revised version of our preprint [FP′]. It extends our
results from finite $T$-CW complexes to a much larger class of $T$-spaces and takes
into account the Goertsches–Töben result.

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paper.

2. The Atiyah–Bredon sequence. We use Alexander–Spanier cohomol-
ogy, cf. [M, Chapter 8]. Coefficients are taken in a subring $R$ of $\mathbb{Q}$ or in the finite
field $R = \mathbb{Z}_p$. Here the letter $p$ denotes a prime, as it does in the rest of the
paper. All topological spaces will be assumed to be paracompact and Hausdorff.
Recall that on locally contractible spaces Alexander–Spanier cohomology and
singular cohomology coincide.

Let $T = (S^1)^n$ be a torus and $X$ a $T$-space. (Note that its Borel construc-
tion $X_T$ is again paracompact Hausdorff.) We assume $X$ to be finite-dimensional,
with only finitely many orbit types and such that $H^*(X^K)$ is finitely generated
over $R$ for all closed subgroups $K$ of $T$.\footnote{These assumptions can be weakened, cf. [AP, Section 3.2]. For example, $X$ can be finitistic
instead of finite-dimensional, and, for $R = \mathbb{Q}$, must only have finitely many connected orbit
types.} For example, $X$ may be a finite $T$-

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CW complex, a compact topological $T$-manifold or a complex algebraic variety
with an algebraic action of $(\mathbb{C}^*)^n \supset T$.

Denote by $X_i, -1 \leq i \leq n$, the equivariant $i$-skeleton of $X$, i.e., the union of
all orbits of dimension \( \leq i \). In particular, \( X_{-1} = \emptyset \), \( X_0 = X^T \) and \( X_n = X \). Since \( X \) is Hausdorff with only finitely many orbit types, each \( X_i \) is closed in \( X \). The maximal \( p \)-torus \( T_p \subset T \) is the subgroup of all elements of order \( p \) (together with \( 1 \in T \)). It is isomorphic to \( \mathbb{Z}_p \). The \( T_p \)-equivariant \( i \)-skeleton \( X_{p,i} \) of \( X \) is the set of all \( T_p \)-orbits consisting of at most \( p^i \) points. One clearly has \( X_i \subset X_{p,i} \).

The inclusion of pairs \((X_i, X_{i-1}) \hookrightarrow (X, X_{i-1})\) gives rise to a long exact sequence
\[
\cdots \to H^*_T(X, X_i) \to H^*_T(X, X_{i-1}) \to H^*_T(X_i, X_{i-1}) \xrightarrow{\delta} H^{*+1}_T(X, X_i) \to \cdots,
\]
and likewise \((X_{i+1}, X_{i-1}) \hookrightarrow (X_{i+1}, X_i)\) gives maps
\[
H^*_T(X_i, X_{i-1}) \to H^{*+1}_T(X_{i+1}, X_i).
\]

**Theorem 2.1.** Let \( X \) be a \( T \)-space such that \( H^*_T(X) \) is a free \( H^*(BT) \)-module. Suppose in addition that for all \( i \)
\[
(2.3a) \quad X_{p,i} = X_i
\]
if \( R = \mathbb{Z}_p \), or, in case \( R \subset \mathbb{Q} \),
\[
(2.3b) \quad X_{p,i-1} \subset X_i
\]
for all \( p \) not invertible in \( R \). Then the following “Atiyah–Bredon sequence” is exact:
\[
(2.4) \quad 0 \to H^*_T(X) \to H^*_T(X_0) \to H^{*+1}_T(X_1, X_0) \to \cdots
\]
\[
\cdots \to H^{*+n-1}_T(X_{n-1}, X_{n-2}) \to H^{*+n}_T(X_n, X_{n-1}) \to 0.
\]

More generally, if condition \((2.3a)\) or \((2.3b)\) holds for all \( i \leq k \), then one still has exactness for
\[
(2.5) \quad 0 \to H^*_T(X) \to H^*_T(X_0) \to H^{*+1}_T(X_1, X_0) \to \cdots
\]
\[
\cdots \to H^{*+k-1}_T(X_{k-1}, X_{k-2}) \to H^{*+k}_T(X_k, X_{k-1}).
\]

The content of the Chang–Skjelbred lemma is just exactness of the Atiyah–Bredon sequence at the first terms. Rephrasing Theorem 2.1 for \( k \in \{0,1\} \) and integer coefficients, we get:

**Corollary 2.2.** Assume \( R = \mathbb{Z} \) and let \( X \) be a \( T \)-space such that \( H^*_T(X) \) is free over \( H^*(BT) \). Then
\[
0 \to H^*_T(X) \to H^*_T(X_0)
\]
is exact. If in addition the isotropy group of each \( x \notin X_1 \) is contained in a proper subtorus of \( T \) (i.e., if condition (2.3b) holds for \( i \leq 1 \)), then the following sequence is exact:

\[
0 \rightarrow H^*_T(X) \rightarrow H^*_T(X_0) \rightarrow H^{*+1}_T(X_1, X_0).
\]

The injectivity of the map \( H^*_T(X) \rightarrow H^*_T(X_0) \) is an immediate consequence of the localization theorem in equivariant cohomology, see Lemma 4.4 below.

The above “proper subtorus condition” is violated in the counterexample given by Tolman–Weitsman [TW, Section 4] because there all orbits of dimension 2 have isotropy group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \subset S^1 \times S^1 \). The full condition (2.3b) can be stated analogously: For \( R = \mathbb{Z} \) the isotropy group \( T_x \) of any \( x \in X \) must be contained in a subtorus of dimension \( \leq \dim T_x + 1 \).

**Remark 2.3.** In some cases general criteria guarantee the freeness of \( H^*_T(X) \) over \( H^*(BT) \), for instance if \( H^*(X) \) is free over \( R \) and concentrated in even degrees (Leray–Hirsch). Assume that \( H^*(X) \) is free over \( R \subset \mathbb{Q} \). Then \( H^*_T(X) \) is free over \( H^*(BT) \) if \( X \) is a compact Kähler manifold with \( X^T \neq \emptyset \) (for example, a smooth projective \( (\mathbb{C}^*)^n \)-variety) [Bl, Theorem II.1.2] or if \( X \) is a compact Hamiltonian \( T \)-manifold [F]. If all isotropy groups of \( X \) are connected, then the exactness of the Atiyah–Bredon sequence with integer coefficients can be characterized by a homological condition on \( H^*_T(X) \), see [FP].

### 3. Isotropy groups.

Before proving Theorem 2.1, we want to reformulate the conditions (2.3a) and (2.3b) on the equivariant skeletons in algebraic terms. In the sequel, we will refer to (2.3a) and (2.3b) together as condition (2.3). Notice that this condition is always satisfied for \( R = \mathbb{Q} \).

For any closed subgroup \( T' \subset T \) there is an isomorphism \( T \cong (S^1)^n \) and a divisor chain \( m_q | m_{q-1} | \cdots | m_1 \) such that \( T' \) corresponds to the subgroup

\[
Z_{m_1} \times \cdots \times Z_{m_q} \times (S^1)^r \subset (S^1)^n.
\]

(To see this, consider the inverse image of \( T' \) under the exponential map \( \exp: t \rightarrow T \) and compare it with the lattice \( \exp^{-1}(1) \).)

Recall that the dimension of a module \( M \) over a ring \( S \) is the Krull dimension of \( S/\text{Ann} M \), that is, the length of a maximal chain \( p_0 \subsetneq p_1 \subsetneq \cdots \) of prime ideals in \( S \), all containing the annihilator \( \text{Ann} M \) of \( M \). We denote the dimension of \( R \) over itself by \( d \). Hence, \( d = 0 \) if \( R \) is a field, and \( d = 1 \) if \( R \subset \mathbb{Q} \). Since \( R \) is a PID, the dimension of a polynomial ring \( R[t_1, \ldots, t_s] \) is \( d + s \) [S, Proposition III.13].

**Lemma 3.1.** Let \( T' \subset T \) be a closed subgroup and let \( s \) be the number of \( m_j \)'s in the decomposition (3.1) which are not invertible in \( R \). Then

\[
\dim_{H^*(BT)} H^*(BT') = \begin{cases} 
  r + s & \text{if } R \text{ is a field or } s > 0, \\
  r + 1 & \text{if } R \subset \mathbb{Q} \text{ and } s = 0.
\end{cases}
\]
Proof. The $H^*(BT)$-action on $H^*(BT')$ comes from the algebra map induced by the inclusion $T' \hookrightarrow T$. Hence, the dimension of the $H^*(BT)$-module $H^*(BT')$ is equal to the Krull dimension of the (evenly graded) image $A$ of $H^*(BT)$ in $H^*(BT')$. We choose isomorphisms $H^{2*}(BS^1) \cong R[t]$ and $H^{2*}(B\mathbb{Z}_m) \cong R[t]/mt$ such that the map $H^{2*}(BS^1) \to H^{2*}(B\mathbb{Z}_m)$ corresponds to the canonical quotient map $R[t] \to R[t]/mt$. By the Künneth theorem for cohomology, we find

$$A \cong H^{2*}(B\mathbb{Z}_m) \otimes \cdots \otimes H^{2*}(B\mathbb{Z}_q) \otimes H^{2*}(BS^1) \otimes R$$

$$\cong R[t_1, \ldots, t_q, t_1', \ldots, t_r']/(m_1 t_1, \ldots, m_q t_q)$$

$$= R[t_1, \ldots, t_s, t_1', \ldots, t_r']/(m_1 t_1, \ldots, m_s t_s)$$

since $m_1, \ldots, m_q$ are invertible in $R$.

$R$, a field or $s = 0$: then $A \cong R[t_1, \ldots, t_s, t_1', \ldots, t_r']$, so the assertion is clear.

$R \subseteq \mathbb{Q}$ and $s > 0$: An ascending chain of prime ideals in $A$ of length $k$ is the same as an ascending chain $p_0 \subset \cdots \subset p_k$ of prime ideals in $B = R[t_1, \ldots, t_s, t_1', \ldots, t_r']$ with $(m_1 t_1, \ldots, m_s t_s) \subset p_0$. Since one can clearly find such a chain of length $r + s$, we have to show that there are no longer chains. If $p_0$ contains some prime factor $p$ of $m_s$, then $B/p_0$ is a quotient of

$$\mathbb{Z}_p[t_1, \ldots, t_r'],$$

which is of dimension $r + s$. Otherwise, $t_s \in p_0$ and $B/p_0$ is a quotient of $R[t_1, \ldots, t_{s-1}, t_1', \ldots, t_r']$, which by induction has again dimension $r + s$. In any case, we find that $k \leq r + s$. \hfill \Box

**Proposition 3.2.** The conditions (2.3) for all $i \leq k$ are equivalent to the conditions

$$\forall x \in X \text{ dim}_{H^*(BT)} H^*(BT_x) \geq d + n - i \quad \Rightarrow \quad x \in X_i$$

for all $i \leq k$.

Proof. $R = \mathbb{Q}$: Since the dimension of $H^*(BT_x)$ is just the rank of $T_x$, condition (3.2) holds for all spaces, as does (2.3).

$R = \mathbb{Z}_p$: By Lemma 3.1, condition (3.2) means that the rank of the maximal $p$-torus contained in $T_x$ equals the dimension of $T_x$ for $x \in X_k$, and differs by at most $i - k - 1$ for $x \in X_i \setminus X_{i-1}$, $i > k$. This is equivalent to $X_{p,i} = X_i$ for $i \leq k$.

$R \subseteq \mathbb{Q}$: Here condition (3.2) translates into the following: for each non-invertible prime $p$ the rank of the maximal $p$-torus in $T_x$ differs by at most 1 from the dimension of $T_x$ for $x \in X_k$, and by at most $i - k$ for $x \in X_i \setminus X_{i-1}$, $i > k$. This is the same as saying $X_{p,i} \subseteq X_i$ for all $i \leq k$. \hfill \Box

Note that conditions (3.2) are always true in $K$-theory (see [A, (7.1)]), essentially because the representation ring of a finite group is a finitely generated $\mathbb{Z}$-module.
4. **Proof of Theorem 2.1.** Before starting in earnest, we remark that our assumptions on $X$ imply that all $R$-modules $H^*(X_j, X_i)$, $j \geq i$, are finitely generated, hence so are the $H^*(BT)$-modules $H^*_T(X_j, X_i)$. (Use the usual long exact sequences for the first claim. For equivariant cohomology, the proof given in [AP, (3.10.1)] for $R = \mathbb{Q}$ still applies.)

The sequences (2.4) and (2.5) are exact if and only if for all $i \geq 0$ (respectively $0 \leq i \leq k$) the sequence (2.1) splits into short exact sequences

\[(4.1) \quad 0 \to H^*_T(X, X_{i-1}) \to H^*_T(X_i, X_{i-1}) \to H^*_{T+1}(X, X_i) \to 0,\]

i.e., if and only if the inclusion of pairs $(X, X_{i-1}) \to (X, X_i)$ induces the zero map in cohomology. (See [FP, Lemma 4.1].)

Let $M$ be a graded $H^*(BT)$-module. Then any prime ideal associated with $M$ is homogeneous, i.e., generated by its homogeneous elements [BH, Lemma 1.5.6], hence contained in some maximal homogeneous ideal $m$. This implies $M = 0$ if and only if $M_m = 0$ for all maximal homogeneous ideals $m \subset H^*(BT)$. It suffices therefore to prove exactness after localizing at such an ideal $m$. Note that it always contains $H^{>0}(BT)$. From now on all localized modules will be over the regular local ring $A = H^*(BT)_m$, whose maximal ideal we denote by $m_A$.

By induction on $i \geq 0$ (resp. $0 \leq i \leq k$), we show:

**Claim 4.1.** The sequence (4.1) is exact, and $H^*_T(X, X_i)_m$ is zero or a Cohen–Macaulay module of dimension $d + n - i - 1$.

Recall that the **depth** of a finitely generated $A$-module $M$ is the maximal length of an $M$-regular sequence in the maximal ideal $m_A$. An $M$-regular sequence is a sequence $a_1, \ldots, a_l \in m_A$ such that $(a_1, \ldots, a_l)M \neq M$ and such that $a_j$ is not a zero divisor on $M/(a_1, \ldots, a_{j-1})M$ for all $j \leq l$. One always has

\[(4.2) \quad \text{depth } M \leq \dim M;\]

if $M \neq 0$ and

\[(4.3) \quad \text{depth } M = \dim M,\]

then $M$ is called **Cohen–Macaulay**. (We use [S] and [BH] as references for commutative algebra. The reader might also find the summary of results in [AP, Appendix A] helpful. They were compiled with applications in equivariant cohomology in mind.)

We start the proof with several lemmas. The first two are standard.

**Lemma 4.2.** Let $0 \to U \to M \to N \to 0$ be an exact sequence of $A$-modules. Then

\[\text{depth } N \geq \min\{\text{depth } U - 1, \text{depth } M\}.\]
Proof. Since
\[ \text{depth } M = \inf \{ i : \text{Ext}^i(K, M) \neq 0 \}, \]
where \( K \) denotes the residue field of \( A \), the assertion follows from the long exact sequence
\[ \cdots \rightarrow \text{Ext}^i(K, U) \rightarrow \text{Ext}^i(K, M) \rightarrow \text{Ext}^i(K, N) \rightarrow \text{Ext}^{i+1}(K, U) \rightarrow \cdots . \]
\[ \square \]

Lemma 4.3. Let \( M \) be a Cohen–Macaulay \( A \)-module. If \( U \subset M \) is a non-zero submodule, then \( \dim U = \dim M \).

Proof. Choose a prime ideal \( p \) associated to \( U \), hence also to \( M \). Since \( M \) is Cohen–Macaulay, we have \( \dim A/p = \dim M \) by [S, Proposition IV.13]. On the other hand, \( \dim A/p \leq \dim U \leq \dim M \).

Lemma 4.4. For all \( i \leq k \) one has \( \dim H^*_T(X, X_i) \leq d + n - i - 1 \).

Proof. This follows from the localization theorem in equivariant cohomology: For a prime ideal \( p \subset H^*(BT) \), define \( X^p = \{ x \in X : p \supset \text{Ann} H^*(BT) \} \). The localization theorem asserts that the inclusion \( X^p \hookrightarrow X \) induces an isomorphism
\[ H^*_T(X)_p = H^*_T(X^p)_p, \]
see for example [AP, Section 3.2]. An analogous formula holds for relative cohomology.

Now assume that \( \dim H^*_T(X, X_i) \geq d + n - i \). This means that there is a prime ideal \( p \supset \text{Ann} H^*_T(X, X_i) \) such that \( \dim H^*(BT)/p \geq d + n - i \). Hence \( \dim H^*(BT) \geq d + n - i \) for any \( x \in X^p \). By Proposition 3.2, this implies \( x \in X_i \), i.e., \( X^p \subset X_i \). The localization theorem now gives
\[ H^*_T(X, X_i)_p = H^*_T(X^p, X^p)_p = H^*_T(X^p, X^p)_p = 0. \]
But this is impossible, because for any prime ideal \( p \) one has \( p \supset \text{Ann} H^*_T(X, X_i) \) if and only if \( H^*_T(X, X_i)_p \neq 0 \) [S, Proposition I.3]. (Here we are using that \( H^*_T(X, X_i) \) is finitely generated over \( H^*(BT) \).) Hence \( \dim H^*_T(X, X_i) \leq d + n - i - 1 \).

Lemma 4.5. For all \( i \geq 0 \) one has depth \( H^*_T(X_i, X_{i-1})_m \geq n - i \).

Proof. Each \( x \in X_i \setminus X_{i-1} \) is fixed by exactly one \((n - i)\)-dimensional torus \( T_\alpha \subset T \). Because \( X \) has only finitely many orbit types, we can write \( X_i \setminus X_{i-1} \) as the disjoint union of finitely many subsets \( Y_\alpha = (X_i \setminus X_{i-1})^{T_\alpha} \). Since \( Y_\alpha \subset Y_\alpha \cup X_{i-1} \), a Mayer–Vietoris argument gives
\[ H^*_T(X_i, X_{i-1}) = \bigoplus_\alpha H^*_T(Y_\alpha, Y_\alpha \cap X_{i-1}). \]
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(Recall that with respect to Alexander–Spanier cohomology, the Mayer–Vietoris sequence is exact for all pairs of closed subspaces, cf. [M, Section 8.6].)

We have

\[ M := H^*_{T}(\bar{Y}_i, \bar{Y}_i \cap X_{i-1}) = H^*_{T/T_{\alpha}}(\bar{Y}_i, \bar{Y}_i \cap X_{i-1}) \otimes_R H^*(BT_{\alpha}) \]

because \( \bar{Y}_i \) is fixed by \( T_{\alpha} \). A sequence of generators \( x_1, \ldots, x_{n-i} \in H^{2}(BT_{\alpha}) \subset m \) clearly is \( M_m \)-regular. Hence

\[ \text{depth } M_m \geq n - i. \]

The description of depth via \( \text{Ext} \) (cf. the proof of Lemma 4.2) shows that the depth of a direct sum is the minimum of the depths of the summands. This gives the result. \( \square \)

We now prove Claim 4.1 by induction on \( i \), assuming that \( H^*_{T}(X, X_{i-1}) \) is zero or a Cohen–Macaulay module of dimension \( d + n - i \). For \( i = 0 \) this is true because we assume \( H^*_{T}(X, X) = H^*_{T}(X) \) to be free over the regular ring \( H^*(BT) \).

In the case \( H^*_{T}(X, X_{i-1}) = 0 \) the sequence (2.1) splits into short exact sequences for trivial reasons. Suppose therefore that \( H^*_{T}(X, X_{i-1}) \) is non-zero and Cohen–Macaulay of dimension \( d + n - i \). Since

\[ \text{dim } H^*_{T}(X, X_i) \leq \text{dim } H^*_{T}(X, X_i) \leq d + n - i - 1 \]

by Lemma 4.4 and \( \text{dim } H^*_{T}(X, X_{i-1}) = d + n - i \), the image of \( H^*_{T}(X, X_i) \) in \( H^*_{T}(X, X_{i-1}) = 0 \) has lower dimension than the target module, hence is zero by Lemma 4.3. In other words, (4.1) is exact in this case as well.

Lemma 4.2 now gives

\[ \text{depth } H^*_{T}(X, X_i) \geq d + n - i - 1 \]

since \( \text{depth } H^*_{T}(X, X_{i-1}) \geq n - i \) by Lemma 4.5 and \( \text{depth } H^*_{T}(X, X_{i-1}) = d + n - i \) (or infinity if \( H^*_{T}(X, X_{i-1}) = 0 \)).

Comparing (4.4) and (4.5) with the inequality (4.2) finishes the proof.

5. Actions without fixed points. The Atiyah–Bredon sequence (2.4) can never be exact if \( T \) acts without fixed points (unless \( X = \emptyset \)). Yet there are interesting fixed-point-free actions. In the context of compact differentiable \( T \)-manifolds and real coefficients, Goertsches and Töben [GT] observed that one can modify (2.4) to accommodate for this case. We briefly comment on such a generalization in our setting.

Proposition 5.1. Let \( k \) be the minimal dimension of the \( T \)-orbits in \( X \). If \( X \) satisfies condition (2.3), then \( \text{dim } H^*_{T}(X) = d + n - k \).
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**Proof.** Since $X_{k-1} = \emptyset$, we have $\dim H^*_T(X) \leq d + n - k$ by Lemma 4.4.

For the reverse inequality, pick an $x \in X_k$, say with isotropy group $T'$. The map $H^*(BT) \to H^*(BT')$ factors through the map $H^*_T(X) \to H^*_T(Tx) = H^*(BT')$ induced by the inclusion $Tx \to X$. This implies $\text{Ann} H^*_T(X) \subset \text{Ann} H^*(BT')$, hence $\dim H^*_T(X) \geq \dim H^*(BT')$. But, under condition (2.3), Lemma 3.1 gives $\dim H^*(BT') = d + n - k$.

A finitely generated graded module $M$ over $H^*(BT)$ is called Cohen–Macaulay if $M_m$ is Cohen–Macaulay over $H^*(BT)_m$ for all maximal ideals $m \in \text{Supp} M$. If $M$ is free over $H^*(BT)$, then it is Cohen–Macaulay of maximal dimension $d + n$ (cf. the proof of Claim 4.1). We remark in passing that the converse holds as well, even if $R$ is not a field: The Auslander–Buchsbaum formula [BH, Theorem 1.3.3] together with [S, Corollary IV.C.2] implies that $M$ is projective; freeness then follows from the (comparatively easy) analogue of the Quillen–Suslin theorem for finitely generated graded modules over polynomial rings with coefficients in a PID [L, Corollary 4.7].

**Theorem 5.2.** Let $k$ be the minimal dimension of the $T$–orbits in $X$. If $H^*_T(X)$ is Cohen–Macaulay over $H^*(BT)$ and $X$ satisfies condition (2.3), then the following sequence is exact:

$$0 \to H^*_T(X) \to H^*_T(X_k) \to H^*_T(X_{k+1}, X_k) \to \cdots$$

$$\cdots \to H^*_T(X_{n-1}, X_{n-2}) \to H^*_T(X_n, X_{n-1}) \to 0.$$

The only change to the proof of Theorem 2.1 is that the induction now starts at $i = k$, which is the dimension of $H^*_T(X)$ by Proposition 5.1.

**References**


Matthias Franz and Volker Puppe


Department of Mathematics, University of Western Ontario, London, ON N6A 5B7
e-mail: mfranz@uwo.ca

FB Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany
e-mail: volker.puppe@uni-konstanz.de