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Dimension groups and multidimensional continued fractions

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Abstract. We describe a class of dimension groups associated with multidimensional continued fractions and show how a certain property of a continued fraction is reflected in the structure of its dimension group.

Résumé. On décrit une classe de groupes de dimensions associés aux fractions continues multidimensionnelles et on montre comment une certaine propriété d'une fraction continue se reflète dans la structure de son groupe de dimensions.

1. **Introduction.** A simplicial group is a direct sum of copies of the integers, viewed as an ordered group with the natural ordering. A dimension group is an inductive limit of simplicial groups. The class of ultrasimplicial groups is an interesting subclass of the class of dimension groups. A dimension group is ultrasimplicial if it can be written as an inductive limit of simplicial groups in which the embedding maps are eventually all injective. The question of which dimension groups are ultrasimplicial has been studied for almost as long as have dimension groups themselves; indeed, in [4], Elliott showed that all totally ordered dimension groups are ultrasimplicial, and that the class of ultrasimplicial groups is closed under various natural operations.

Elliott's argument about totally ordered groups was essentially a modified version of the continued fraction algorithm for approximating real numbers with rational numbers. Recognizing this, Effros and Shen used the continued fraction algorithm to classify the simple free dimension groups of rank two in [3]. These groups are precisely the subgroups of the real numbers with the form $\mathbb{Z} + \mathbb{Z}\alpha$, where α is an irrational number. Then in [2], Effros and Shen found an explicit inductive limit representation, using injective maps, of the dimension group $\mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are rationally independent. Their proof used a multidimensional variant of the continued fraction algorithm called the Jacobi–Perron algorithm.

To date, various classes of dimension groups have been shown to be ultrasimplicial (see [4], [3], [2], [6] and [5]); of these proofs, only the one appearing in [5] does not rely on a variant of the continued fraction algorithm. This suggests a

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natural question: do all free finite rank ultrasimplicial groups arise from continued fractions? If not, what can be said about those groups that do arise from continued fractions? This is the question considered here.

2. Continued fractions and continued fraction groups. Fix some $n \geq 2$, let l_0 be a vector in $\mathbb{R}^n$ with non-negative entries, and let $l = \mathbb{R}^+ l_0$ denote the ray that it generates. Let $\big(I(k)\big)_{k=1}^{\infty}$ be a sequence of $n \times n$ matrices, let $A(0)$ denote the $n \times n$ identity matrix, and for $k > 0$ let $A(k)$ denote the product $I(k)A(k-1)$. Then let us call $\big(I(k)\big)_{k=1}^{\infty}$ a multidimensional continued fraction expansion of l if for all k the following conditions are satisfied:

(1) $I(k)$ is a transvection matrix; that is, all of its diagonal entries are 1 and all of its off-diagonal entries are 0 except for the entry at position (s_k, t_k), which is a positive integer; let us call it b_k. Let us denote the transvection matrix that has the number b at position (s, t) by I_{st}^b.

(2) Letting $A_1(k), \ldots, A_n(k)$ denote the rows of the matrix $A(k)$, we have

$$l_0 \in A(k)^+ := \Big\{ \sum x_j A_j(k) \;\Big|\; x_j \geq 0 \Big\}.$$

(l_0 is in the positive cone generated by $A_1(k), \ldots, A_n(k)$.)

This definition appears in [1], along with an explanation of why it can be viewed as a multidimensional generalization of the ordinary continued fraction algorithm.

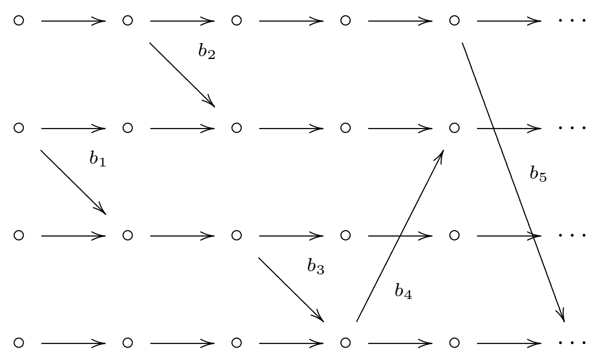
Let $E = (I(k))_{k=1}^{\infty}$ be a continued fraction expansion of some ray $l \subset \mathbb{R}^n$. We can associate a dimension group G_E to E in the following way: let G_E be the limit of the sequence

$$\mathbb{Z}^n \xrightarrow{\varphi_1} \mathbb{Z}^n \xrightarrow{\varphi_2} \mathbb{Z}^n \xrightarrow{\varphi_3} \cdots$$

which we will call the standard inductive limit representation of G_E. Here $\varphi_k \colon \mathbb{Z}^n \to \mathbb{Z}^n$ is the map given by left multiplication by the matrix $I(k)$. Let us call G_E the dimension group of E, and let us refer to the class of all such groups G_E as the class of continued fraction groups. Let us also include dimension groups G_E arising from finite sequences $E = \big(I(k)\big)_{k=1}^{N}$ in this class. If E is a finite sequence, then the group G_E is order isomorphic to the simplicial group $\mathbb{Z}^n$.

3. Closure of the class of continued fraction groups under natural operations. Recall that an ideal of a dimension group corresponds to a subset of its Bratteli diagram (any Bratteli diagram representing the group will work) that is hereditary and directed. A subset D of a Bratteli diagram is hereditary if it has the property that, if v is a node in D and there is an arrow from v to another node w, then w is also in D. To say that D is directed means that, whenever D contains all of the nodes w for which there is an arrow from v to w, D also contains v.

In the Bratteli diagram for the standard inductive limit representation of a continued fraction group there are exactly n nodes for each subgroup $\mathbb{Z}^n$. For the k-th subgroup, call these n nodes $(k,1),\ldots,(k,n)$. Then there is an arrow with multiplicity 1 from (k,j) to $(k+1,j)$ for all j, as well as a single arrow with multiplicity b_k from (k,t_k) to $(k+1,s_k)$. In other words, the diagram looks something like this:



It is not hard to check that an ideal of a continued fraction group must arise from a subset S of $\{1,\ldots,n\}$ with the property that, beyond a certain index k_0, there is no arrow with its beginning t_k in S and its end s_k out of S. By dropping to a subsequence (which does not change the inductive limit), we may suppose that $k_0 = 1$. Then the ideal is the inductive limit of the subdiagram containing the nodes $\{(k,j) \mid k \in \mathbb{N}, j \in S\}$ and the arrows that both begin and end at these nodes.

The inductive limit structure of such a subdiagram is exactly that of a continued fraction group. The embedding of the k-th subgroup into the $(k+1)$-st is given by left multiplication by the submatrix obtained from $I(k)$ by removing all rows and columns with indices not in S. The result of removing simultaneously the j-th row and the j-th column from an $n\times n$ transvection matrix is either an $(n-1)\times(n-1)$ transvection matrix or the $(n-1)\times(n-1)$ identity matrix. In either case, the new inductive limit decomposition is that of another continued fraction group.

The situation for quotients of continued fraction groups by ideals is exactly the same. One may obtain a diagram for the quotient of a dimension group by an ideal simply by taking a diagram for the original dimension group and removing the subdiagram corresponding to the ideal, as well as any arrows that end in this subdiagram. Then the diagram for the quotient of a continued fraction group by an ideal contains exactly those nodes not in the set S that defines the ideal. By the argument used above, such a diagram has the inductive limit structure of another continued fraction group.

This discussion results in the following theorem.

THEOREM 1. *The class of continued fraction groups is closed under passing to ideals and quotients by ideals.*

Next let us consider direct sums and lexicographic direct sums of continued fraction groups.

Recall that, if (G, G^+) and (H, H^+) are ordered groups, then their ordered group direct sum, denoted $G \oplus H$, has the abelian group $G \oplus H$ as its underlying group, with order defined by the positive cone $\{(g, h) \mid g \geq 0, h \geq 0\}$. The lexicographic direct sum, denoted $G \oplus_{\text{lex}} H$, also has $G \oplus H$ as the underlying group, but the positive cone is $\{(g, h) \mid g > 0\} \cup \{(0, h) \mid h \geq 0\}$.

THEOREM 2. *If (G, G^+) and (H, H^+) are continued fraction groups, then so is $G \oplus_{\text{lex}} H$.*

PROOF. Let us prove the theorem by constructing for the lexicographic direct sum a Bratteli diagram that is obviously the diagram of a continued fraction group.

Write $G = \varinjlim(G_k, \phi_k)$ and $H = \varinjlim(H_k, \psi_k)$, where $G_k = \mathbb{Z}^{n_1}$ and $H_k = \mathbb{Z}^{n_2}$ for all k.

Denote by $\big(I(k)\big)_{k=1}^{\infty}$ and $\big(J(k)\big)_{k=1}^{\infty}$ the continued fractions that give rise to G and H respectively (that is, the maps ϕ_k and ψ_k arise from left multiplication by $I(k)$ and $J(k)$ respectively). We may suppose by breaking the inductive limit into smaller steps that $I(k)$ and $J(k)$ are transvection matrices in which the only non-zero off-diagonal entry is 1. This is because every transvection matrix is a power of such matrices (that is, $I_{st}^{b} = (I_{st}^{1})^{b}$).

Then consider the following diagram:

$$
\begin{array}{ccccccccccccccc}
G_1 & \xrightarrow{\phi_1} & G_2 & \xrightarrow{\text{id}} & G_2 & \xrightarrow{\text{id}} & G_2 & \xrightarrow{\phi_2} & G_3 & \xrightarrow{\text{id}} & G_3 & \xrightarrow{\text{id}} & G_3 & \cdots \\
 & & & & & \searrow^{\rho_1} & & & & & & \searrow^{\rho_2} & & \\
H_1 & \xrightarrow[\text{id}]{} & H_1 & \xrightarrow[\psi_1]{} & H_2 & \xrightarrow[\text{id}]{} & H_2 & \xrightarrow[\text{id}]{} & H_2 & \xrightarrow[\psi_2]{} & H_3 & \xrightarrow[\text{id}]{} & H_3 & \cdots
\end{array}
$$

Here, the map ρ_k is given by left multiplication by the $n_2 \times n_1$ matrix the entries of which are all 2^k. Then the matrix of partial embeddings for the step

$$
\begin{array}{ccc}
G_k & \xrightarrow{\text{id}} & G_k \\
 & \searrow^{\rho_k} & \\
H_k & \xrightarrow[\text{id}]{} & H_k
\end{array}
$$

is the $(n_1+n_2)\times(n_1+n_2)$ matrix

$$(*)\qquad \left[\begin{array}{ccc|c} & I_{n_1} & & 0 \\ \hline 2^k & \cdots & 2^k & \\ \vdots & & \vdots & I_{n_2} \\ 2^k & \cdots & 2^k & \end{array}\right] = \prod_{j\le n_1<i} I_{ij}^{2^k},$$

which is clearly a product of transvection matrices.

Since ϕ_k and ψ_k can be represented by transvection matrices, so too can the embeddings $\phi_k\oplus\mathrm{id}$ and $\mathrm{id}\oplus\psi_k$. Therefore the given diagram represents a continued fraction group; let us call it F. We can write F as a direct limit of the groups $G_k\oplus H_k\cong\mathbb{Z}^{n_1+n_2}$ under the maps θ_k, where θ_k is the composition of the three maps $\phi_k\oplus\mathrm{id}$, $\mathrm{id}\oplus\psi_k$, and the third map with the embedding matrix $(*)$. It remains to be shown that F is indeed the lexicographic direct sum.

H is an ideal of F, and G is the quotient, so as a group $F\cong G\oplus H$ (indeed, an ideal of any dimension group is a group direct summand, although not necessarily an ordered group direct summand). Let $\theta_{k\infty}\colon F_k\to F$, $\phi_{k\infty}\colon G_k\to G$, and $\psi_{k\infty}\colon H_k\to H$ denote the maps into the inductive limit. Then, because ϕ_k,ψ_k, and θ_k are all group isomorphisms, so too are $\phi_{k\infty},\psi_{k\infty}$, and $\theta_{k\infty}$. Then

$$G\oplus H \xrightarrow{\phi_{1\infty}^{-1}\oplus\psi_{1\infty}^{-1}} G_1\oplus H_1 \xrightarrow{\mathrm{id}} F_1 \xrightarrow{\theta_{1\infty}} G\oplus H$$

is an isomorphism of groups; let us denote it by γ. We will show that, when $G\oplus H$ is given the lexicographic ordering, γ is in fact an isomorphism of ordered groups.

Pick $(g,h)\in G\oplus H$. We must test three cases: (1) $g\notin G^+$, (2) $g=0$, and (3) $g>0$.

(1) Suppose $g\notin G^+$. Then, letting $g_k=\phi_{k\infty}^{-1}(g)$, $g_k\notin G_k^+$ for all k. But g_k is the first coordinate of $\theta_{k\infty}^{-1}\circ\gamma(g,h)$, so $\gamma(g,h)$ is non-positive in every subgroup $F_k=G_k\oplus H_k$ of F, and so is non-positive in F itself.

(2) Suppose $g=0$ and let $h_k=\psi_{k\infty}^{-1}(h)$. Then, because $g=0$, the second coordinate of $\theta_{k\infty}^{-1}\circ\gamma(g,h)$ is h_k. If $h\in H^+$, then eventually $h_k\in H_k^+$, and so $(0,h_k)\in F_k^+=G_k^+\oplus H_k^+$. If $h\notin H^+$, then h_k is not in H_k^+ for any k, and so $(0,h_k)\notin F_k^+$ for any k. Therefore $\gamma(0,h)\in F^+$ if and only if $h\in H^+$.

(3) Now suppose that $g>0$ and $h\in H$. We wish to show that $\gamma(g,h)\in F^+$. Since $g\in G^+$, there exists some index k_0 such that, for all $k>k_0$, $g_k>0$ (that is, g_k is a vector of non-negative integers, at least one of which is strictly positive). Let $h'_k\in H_k$ be such that $\theta_{k\infty}^{-1}\circ\gamma(g,h)=(g_k,h'_k)$. (In general, $h'_k\neq h_k$ because of the contributions from the maps ρ.)

If h'_{k_0} has all positive entries, then we are done, so we may suppose that $M < 0$ is the least entry of h'_{k_0}. Then the vector $\psi_{k_0} h'_{k_0}$ has least entry no less than $2M$ (recall that the only non-zero off-diagonal entry of the transvection matrix $J(k_0)$ is $b = 1$). Since g_{k_0} has all non-negative entries and at least one strictly positive entry, the entries of $\rho_{k_0} g_{k_0}$ are all greater than or equal to 2^{k_0}. Thus the least entry of $h'_{k_0+1} = \psi_{k_0} h'_{k_0} + \rho_k g_{k_0}$ is no less than

$$2M + 2^{k_0} = 2(M + 2^{k_0-1}).$$

Likewise, the least entry of h'_{k+2} is no less than

$$2\big(2(M + 2^{k_0-1})\big) + 2^{k_0+1} = 4(M + 2^{k_0}).$$

Continuing in this fashion, we see that the least entry of h'_{k_0+j} is no less than

$$2^j(M + 2^{k_0+j-2}),$$

which is eventually strictly positive. Therefore, eventually, $(g_{k_0+j}, h'_{k_0+j}) \in F^+_{k_0+j}$, and so $(g,h) \in F^+$. $\square$

Note that a very similar argument, using the diagram

$$G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\text{id}} G_2 \xrightarrow{\phi_2} G_3 \xrightarrow{\text{id}} G_3 \xrightarrow{\phi_3} \cdots$$

$$H_1 \xrightarrow{\text{id}} H_1 \xrightarrow{\psi_1} H_2 \xrightarrow{\text{id}} H_2 \xrightarrow{\psi_2} H_3 \xrightarrow{\text{id}} \cdots$$

can be used to prove the following proposition.

THEOREM 3. *If (G, G^+) and (H, H^+) are continued fraction groups, then so is the ordered group direct sum $G \oplus H$.*

4. **Convergence.**

There are two basic notions of convergence for continued fractions, called strong convergence and weak convergence. Let $E = \big(I(k)\big)_{k=1}^{\infty}$ be a continued fraction of a ray $l \subset \mathbb{R}^n$. Most sources define strong convergence by saying that E is strongly convergent if the rows of the matrices $A(k) = I(k)I(k-1)\cdots I(1)$ converge to l as $k \to \infty$. Let us use a slightly weaker definition: let us say that E is strongly convergent if, for all $j \le n$, the sequence of row vectors $\big(A_j(k)\big)_{k=1}^{\infty}$ of $A(k)$ admits a subsequence that converges to l.

Let us say that E is weakly convergent (or, alternatively, directionally convergent), if the rows of the matrices $A(k)$ tend to l directionally; that is,

$$\lim_{k\to\infty} \frac{A_j(k)}{\|A_j(k)\|} = \frac{l_0}{\|l_0\|}$$

for all $j \leq n$.

It is natural to wonder if these notions of convergence can be expressed in terms of the dimension group G_E. The answer turns out to be "no" for strong convergence and "yes" for weak convergence.

Weak convergence of E corresponds to a property of the state space of G_E. Recall that a state of G_E is a normalized positive linear functional on G_E (that is, a positive map from G_E to $\mathbb{R}$ that is normalized on an element $u \in G_E$, called an order unit). Then weak convergence has the following meaning in terms of dimension groups:

THEOREM 4. *A continued fraction $E = \big(I(k)\big)_{k=1}^{\infty}$ is weakly convergent if and only if the corresponding dimension group G_E admits a unique state.*

PROOF. The proof relies on the fact that the sequence $A(1)^+ \supset A(2)^+ \supset A(3)^+ \supset \cdots$ is a decreasing sequence of cones, and E is weakly convergent if and only if the intersection of this decreasing sequence is equal to the ray l.

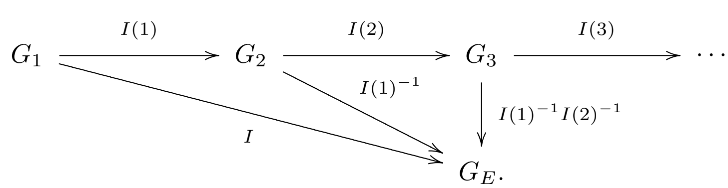
We will prove the theorem by showing that, as a cone, the set of linear functionals on G_E is equal to $\bigcap_k A(k)^+$. Then G_E has a unique normalized positive functional if and only if the cone of positive functionals is singly generated (*i.e.*, it is a ray), which happens if and only if $\bigcap_k A(k)^+ = l$, which happens if and only if E is weakly convergent.

Let τ be a linear functional on G_E. Then we may view τ as a row vector in $(\mathbb{R}^n)^*$.

As groups, G_k and G_E are both isomorphic to $\mathbb{Z}^n$, and the inductive limit map $\phi_{k\infty}\colon G_k \to G_E$ is given by left multiplication by

$$A(k-1)^{-1} = I(1)^{-1} I(2)^{-1} \cdots I(k-1)^{-1}.$$

Indeed, these maps make the following diagram commutative:



Let $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. Then $A(k-1)^{-1}e_1, \dots, A(k-1)^{-1}e_n$ generate the positive cone of the image of G_k in G_E. Therefore τ is positive on the image of G_k in G_E if and only if

$$\tau A(k-1)^{-1} e_j \geq 0$$

for all $j \leq n$.

We can write τ as a linear combination of the rows of $A(k-1)$:

$$\tau = a_1 A_1(k-1) + \cdots + a_n A_n(k-1).$$

But the row vector $A_j(k-1)$ is equal to $e_j^t A(k-1)$, so

$$\tau = a_1 e_1^t A(k-1) + \cdots + a_n e_n^t A(k-1) = (a_1 e_1^t + \cdots + a_n e_n^t) A(k-1).$$

Then τ is positive on the image of G_k in G_E if and only if, for all $j \leq n$,

$$\begin{gathered} \tau A(k-1)^{-1} e_j \geq 0 \\ (a_1 e_1^t + \cdots + a_n e_n^t) A(k-1) A(k-1)^{-1} e_j \geq 0 \\ (a_1 e_1^t + \cdots + a_n e_n^t) e_j \geq 0 \\ a_j \geq 0. \end{gathered}$$

In other words, τ is positive on the image of G_k in G_E if and only if τ lies in $A(k-1)^+$. Therefore a functional τ on G_E is positive on G_E if and only if it lies in the intersection $\bigcap_k A(k)^+$. □

The following example shows that, even in dimension $n = 3$, the isomorphism class of the continued fraction group G_E is insufficient to determine if E is strongly convergent; that is, there exists a ray $l \subset \mathbb{R}^3$ and two continued fraction expansions E_1 and E_2 of l such that E_2 is strongly convergent to l, E_1 is not, and $G_{E_1} \cong G_{E_2}$.

EXAMPLE 1. Consider the continued fraction expansion E_1 with replacement matrices

$$I(k) = \begin{cases} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } k \equiv 1 \mod 3, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix} & \text{if } k \equiv 2 \mod 3, \\ \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } k \equiv 0 \mod 3. \end{cases}$$

Define the matrix

$$B := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 6 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then notice that

$$B^3 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the group G_{E_1} is isomorphic to the direct limit

$$\mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^3 \xrightarrow{B} \cdots .$$

The matrix B has characteristic polynomial $f(x) = x^3 - 6x - 1$, which has roots $\alpha_1 \approx -2.36147, \alpha_2 \approx -0.16745$, and $\alpha_3 \approx 2.52892$. The associated left eigenvectors are, respectively,

$$v_1 := [1 \;\; \alpha_1 \;\; \alpha_1^2], \quad v_2 := [1 \;\; \alpha_2 \;\; \alpha_2^2], \quad v_3 := [1 \;\; \alpha_3 \;\; \alpha_3^2].$$

This expansion converges weakly to the line generated by v_3, its left Perron–Frobenius eigenvector. To see this, first write the standard basis vector e_1^t as

$$e_1^t = c_1 v_1 + c_2 v_2 + c_3 v_3.$$

Notice that, since $\alpha_1, \alpha_2, \alpha_3$, and their squares are all pairwise rationally independent, the coefficients c_j are all non-zero.

Recall that E_1 converges weakly to v_3 if the rows of $A(k) = I(k)I(k-1) \cdots I(1)$ converge directionally to v_3. This happens if and only if the rows of B^k converge directionally to v_3, and the rows of B^k are equal to $e_1^t B^k$, $e_2^t B^k$, and $e_3^t B^k$. Moreover,

$$e_1^t B^k = c_1 \alpha_1^k v_1 + c_2 \alpha_2^k v_2 + c_3 \alpha_3^k v_3,$$

so $\|e_1^t B^k\|$ tends to $c_3 \alpha_3^k \|v_3\|$, and so the directional limit is

$$\begin{aligned} \lim_{k\to\infty} \frac{e_1^t B^k}{c_3 \alpha_3^k \|v_3\|} &= \lim_{k\to\infty} \left(\frac{c_1}{c_3} \left(\frac{\alpha_1}{\alpha_3}\right)^k \frac{v_1}{\|v_3\|} + \frac{c_2}{c_3} \left(\frac{\alpha_2}{\alpha_3}\right)^k \frac{v_2}{\|v_3\|} + \frac{c_3}{c_3} \left(\frac{\alpha_3}{\alpha_3}\right)^k \frac{v_3}{\|v_3\|} \right) \\ &= \frac{v_3}{\|v_3\|}, \end{aligned}$$

and likewise for e_2^t and e_3^t.

Recall from the proof of Theorem 4 that the unique state on G_{E_1} is given by left multiplication by v_3. Since $f(x)$ is irreducible, the numbers 1, α_3, and α_3^2 are rationally independent, and so in fact this state is injective, and G_{E_1} is isomorphic to the subgroup $\mathbb{Z} + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_3^2$ of $\mathbb{R}$. Indeed, this is true for any expansion E that is weakly convergent to v_3.

However, the expansion E_1 is not strongly convergent. To see this, notice that

$$e_1^t B^k = c_1 \alpha_1^k v_1 + c_2 \alpha_2^k v_2 + c_3 \alpha_3^k v_3 \approx c_1 \alpha_1^k v_1 + c_3 \alpha_3^k v_3$$

for large k ($|\alpha_2| < 1$), and the vectors $c_1 \alpha_1^k v_1$ grow without bound in the directions $\pm v_1$. Therefore the distance between $e_j^t B^k$ and $\mathbb{R} v_3$ grows without bound as $k \to \infty$.

In [1], it is shown that a vector having rationally independent entries admits an expansion that is strongly convergent in the sense described above. In particular, there is some strongly convergent expansion E_2 of $\mathbb{R}^+ v_3$. Again, the group G_{E_2} is isomorphic to the subgroup $\mathbb{Z} + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_3^2$ of $\mathbb{R}$.

In other words, $G_{E_1} \cong G_{E_2}$, yet E_2 is strongly convergent and E_1 is not.

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