C. R. Math. Rep. Acad. Sci. Canada Vol. **33** (2) 2011, pp. 44–49

Quasitraces are traces: A short proof of the finite-nuclear-dimension case

Nathanial P. Brown and Wilhelm Winter

Presented by George Elliott, FRSC

Abstract. Uffe Haagerup proved that quasitraces on unital exact C^*-algebras are traces. We give a short proof under the stronger hypothesis of locally finite nuclear dimension; our result generalizes to the case of lower semicontinuous extended quasitraces on nonunital C^*-algebras.

Résumé. Uffe Haagerup a démontré qu'une quasi-trace sur une C^*-algèbre exacte à élément unité est une trace. Nous donnons une courte démonstration sous l'hypothèse plus forte de dimension nucléaire localement finie; ce résultat se généralise jusqu'au cas d'une quasi-trace étendue semicontinue inférieurement sur une C^*-algèbre sans élément unité.

In 1991 Uffe Haagerup distributed a handwritten manuscript [3] containing a proof of the following theorem: Every quasitrace on a unital, exact C^*-algebra is a trace.[1] We will give a very short proof[2] for quasitraces on (not necessarily unital) C^*-algebras with locally finite nuclear dimension (in the sense of [9]), a case that covers all simple C^*-algebras that have been classified by their Elliott invariants so far; in fact, at this point we do not know any example of a nuclear C^*-algebra which does not have locally finite nuclear dimension. We generalize this result to lower semicontinuous extended quasitraces on C^*-algebras with locally finite nuclear dimension.

Our first three lemmas are well known, but we isolate them for convenience. The proof of the first one reduces to the fact that a quasitrace on a full matrix algebra must take the same value on all minimal projections (since they are all Murray–von Neumann equivalent). In the following, A and B are C^*-algebras, not necessarily unital.

[1] In this note, "trace" means a positive, linear functional satisfying the trace condition $\tau(ab) = \tau(ba)$ and "quasitrace" is synonymous with "2-quasitrace" as defined in [1] (in particular, a quasitrace is finite everywhere). We use the term "extended trace" (or "extended quasitrace", respectively) when the value $+\infty$ is allowed on the positive cone.

[2] In fact, the first published proof. In [4], it is shown that every state on $K_0(A)$ arises from a tracial state, when A is unital and exact. In [5] it is shown that every simple, stably projectionless, exact C^*-algebra has a densely defined trace. And [2] shows that every quasitrace on a non-unital exact C^*-algebra is a trace, via reduction to Haagerup's unpublished paper [3]. None of this published work covers our result.

Received by the editors on May 12, 2010; revised October 21, 2010.

The first-named author is partially supported by DMS-0856197. The second-named author is partially supported by EPSRC First Grant EP/G014019/1.

AMS Subject Classification: 46L05.

LEMMA 1. *Quasitraces on finite-dimensional C^*-algebras are traces.*

LEMMA 2. *If $\varphi\colon A \to B$ is a completely positive order zero map (*cf. *[8]) and τ is a trace (resp. quasitrace) on B, then $\tau \circ \varphi$ is a trace (resp. quasitrace) on A.*

PROOF. This is Corollary 3.4 in [8]. □

LEMMA 3. *Let τ be a quasitrace on A and assume there is a trace τ' on A such that $\tau \le \tau'$ (*i.e., *$\tau(x) \le \tau'(x)$ for all $x \in A_+$). Then τ is a trace.*

PROOF. This follows from Corollaries II.1.9 and II.2.4 in [1]. □

Here is the main result.

THEOREM 4. *If A has nuclear dimension n, then every quasitrace τ on A is a trace.*

PROOF. By Lemma 3, it suffices to show the existence of a trace τ' on A such that $\tau \le \tau'$.

According to [9, Proposition 3.2] we can find nets of completely positive maps $\psi_\lambda\colon A \to \bigoplus_{i=0}^n F_\lambda^{(i)}$ and $\varphi_\lambda\colon \bigoplus_{i=0}^n F_\lambda^{(i)} \to A$ such that each $F_\lambda^{(i)}$ is finite dimensional; $\|a - \varphi_\lambda \circ \psi_\lambda(a)\| \to 0$ for all $a \in A$; $\varphi_\lambda^{(i)} := \varphi_\lambda|_{F_\lambda^{(i)}}$ is contractive with order zero for all i, λ; $\|\psi_\lambda\| \le 1$ for all λ; and the induced map

$$\bar{\psi}\colon A \to \frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda}$$

has order zero.

Since each $\varphi_\lambda^{(i)}$ has order zero, $\tau \circ \varphi_\lambda^{(i)}$ is a trace on $F_\lambda^{(i)}$ (combine Lemmas 1 and 2). Since τ is bounded [1, Corollary II.2.3], $\sup_\lambda \|\tau \circ \varphi_\lambda^{(i)}\| < \infty$ and hence, for each free ultrafilter ω on the index set Λ, we get a trace $(\tau \circ \varphi_\lambda^{(i)})_\omega$ on

$$\frac{\prod_\lambda F_\lambda^{(i)}}{\bigoplus_\lambda F_\lambda^{(i)}}$$

by taking the limit of the $\tau \circ \varphi_\lambda^{(i)}$'s along ω. Letting

$$\bar{\psi}_i\colon A \to \frac{\prod_\lambda F_\lambda^{(i)}}{\bigoplus_\lambda F_\lambda^{(i)}}$$

be the induced order-zero map, the composition $(\tau \circ \varphi_\lambda^{(i)})_\omega \circ \bar{\psi}_i$ is a trace on A.

The final fact we need is "2-subadditivity" of quasitraces, *i.e.*, the fact that $\tau(x+y) \le 2\big(\tau(x)+\tau(y)\big)$ for all $x, y \in A_+$ (see [1, Corollary II.2.5]). An induction argument shows the (far from sharp) inequality $\tau(\sum_0^n x_i) \leq 2^n \sum_0^n \tau(x_i)$ for positive x_i, so the following completes the proof:

$$\begin{aligned}\tau(x) &= \lim_\omega \tau\big(\varphi_\lambda \circ \psi_\lambda(x)\big)\\ &\leq 2^n \sum_{i=0}^n \lim_\omega \tau\big(\varphi_\lambda^{(i)} \circ \psi_\lambda^{(i)}(x)\big)\\ &= 2^n \sum_{i=0}^n (\tau \circ \varphi_\lambda^{(i)})_\omega \circ \bar{\psi}_i(x).\end{aligned}$$

□

Since quasitraces are norm-continuous [1, Corollary II.2.5], the following generalization is easily deduced.

COROLLARY 5. *If A has locally finite nuclear dimension (*i.e.*, every finite set is nearly contained in a subalgebra that has finite nuclear dimension), then every quasitrace on A is a trace.*

We are indebted to George Elliott for drawing our attention to the question whether the preceding results generalize to extended quasitraces. The answer is affirmative in the lower semicontinuous case.

THEOREM 6. *If A has locally finite nuclear dimension, then every lower semicontinuous extended quasitrace τ on A is an extended trace.*

PROOF. Let us first assume that A has finite nuclear dimension.

It will suffice to show that $\tau(a+b) = \tau(a) + \tau(b)$ for all $a, b \in A_+$ for which $\tau(a)$ and $\tau(b)$ are both finite.

Observe that $a+b$ is Cuntz subequivalent to $a \oplus b$ in the sense of [1]. By [7, Proposition 2.4], for any $\epsilon > 0$ there are $\delta > 0$ and $x \in M_{2,1}(A)$ such that $(a+b-\epsilon)_+ = x^*\big((a-\delta)_+ \oplus (b-\delta)_+\big)x$. By [1] this implies that

$$\begin{aligned}d_\tau\big((a+b-\epsilon)_+\big) &\leq d_\tau\big((a-\delta)_+ \oplus (b-\delta)_+\big)\\ &= d_\tau\big((a-\delta)_+\big) + d_\tau\big((b-\delta)_+\big)\\ &\leq \frac{1}{\delta}\big(\tau(a) + \tau(b)\big)\\ &< \infty,\end{aligned}$$

where d_τ denotes the lower semicontinuous dimension function associated with τ.

Again by [1] for any positive contraction

$$c \in B_\epsilon := \overline{\big((a+b-2\epsilon)_+\big)A\big((a+b-2\epsilon)_+\big)} \subset A$$

we have $\tau(c) \le d_\tau\big((a+b-\epsilon)_+\big) < \infty$, whence τ restricts to a (bounded) quasitrace τ_ϵ on B_ϵ. Since B_ϵ has finite nuclear dimension by [9, Proposition 2.5], we see from Theorem 4 that every τ_ϵ is in fact trace on B_ϵ.

Let $h_\epsilon \in \mathcal{C}_0(0,1]$ be the positive function which is 0 on $(0,2\epsilon]$, 1 on $[3\epsilon,1]$, and linear on $[2\epsilon,3\epsilon]$, and set $d_\epsilon := h_\epsilon(a+b) \in A$; note that the d_ϵ form an approximate unit for $\overline{(a+b)A(a+b)} \subset A$ and that $B_\epsilon = \overline{d_\epsilon A d_\epsilon}$. By lower semicontinuity of τ, and since each τ_ϵ is a trace on B_ϵ, we have

$$
\begin{aligned}
\tau(a+b) &= \lim_{\epsilon\to 0} \tau(d_\epsilon(a+b)d_\epsilon) \\
&= \lim_{\epsilon\to 0} \tau_\epsilon\big(d_\epsilon(a+b)d_\epsilon\big) \\
&= \lim_{\epsilon\to 0} \tau_\epsilon(d_\epsilon a d_\epsilon) + \lim_{\epsilon\to 0} \tau_\epsilon(d_\epsilon b d_\epsilon) \\
&= \lim_{\epsilon\to 0} \tau(d_\epsilon a d_\epsilon) + \lim_{\epsilon\to 0} \tau(d_\epsilon b d_\epsilon) \\
&= \tau(a) + \tau(b),
\end{aligned}
$$

as desired.

Next, suppose A has only locally finite nuclear dimension, and let $a,b \in A$ be positive contractions. Suppose first that $\tau(a)$, $\tau(b)$, $d_\tau(a)$, $d_\tau(b)$ are all finite, say bounded by some $K \ge 1$.

Let $\eta > 0$ and choose $0 < \epsilon < \eta/8K$ such that

$$
\tau\big((a+b-4\epsilon)_+\big) \ge \tau(a+b) - \eta, \quad \tau\big((a-2\epsilon)_+\big) \ge \tau(a) - \eta
$$
$$
\text{and} \quad \tau\big((b-2\epsilon)_+\big) \ge \tau(b) - \eta;
$$

this is possible by lower semicontinuity of τ. Since A has locally finite nuclear dimension, there are a C^*-subalgebra $B \subset A$ with finite nuclear dimension and positive contractions $a', b' \in B$ such that

$$
\|a'-a\|, \|b'-b\| < \epsilon.
$$

From [6, Lemma 2.2] we see that there are contractions $x,y,z,r,s,v,w \in A$ such that

$$
(a'-\epsilon)_+ = x^*ax, \quad (b'-\epsilon)_+ = y^*by, \quad (a'+b'-2\epsilon)_+ = z^*(a+b)z,
$$
$$
(a-2\epsilon)_+ = r^*(a'-\epsilon)_+r, \quad (b-2\epsilon)_+ = s^*(b'-\epsilon)s,
$$
$$
(a+b-4\epsilon)_+ = v^*(a'+b'-2\epsilon)_+v,
$$
$$
\big((a'-\epsilon)_+ + (b'-\epsilon)_+ - 4\epsilon\big)_+ = w^*(a+b)w.
$$

We estimate

$$
\begin{aligned}
\tau(a+b) &\ge \tau\Big(\big((a'-\epsilon)_+ + (b'-\epsilon)_+ - 4\epsilon\big)_+\Big) \\
&\ge \tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big) - 4\epsilon \cdot d_\tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big)
\end{aligned}
$$

$$\begin{aligned}
&\geq \tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big) - 4\epsilon \cdot \Big(d_\tau\big((a'-\epsilon)_+\big) + d_\tau\big((b'-\epsilon)_+\big)\Big)\\
&\geq \tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big) - 4\epsilon \cdot \big(d_\tau(a) + d_\tau(b)\big)\\
&\geq \tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big) - \eta\\
&\geq \tau\big((a'+b'-2\epsilon)_+\big) - \eta\\
&\geq \tau\big((a+b-4\epsilon)_+\big) - \eta\\
&\geq \tau(a+b) - 2\eta,
\end{aligned}$$

which implies

$$\big|\tau(a+b) - \tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big)\big| < \eta.$$

Since $\tau|_B$ is an extended trace by the first part of the proof, we have

$$\tau\big((a'-\epsilon)_+ + (b'-\epsilon)_+\big) = \tau\big((a'-\epsilon)_+\big) + \tau\big((b'-\epsilon)_+\big).$$

Moreover, we have

$$\tau(a) \geq \tau\big((a'-\epsilon)_+\big) \geq \tau\big((a-2\epsilon)_+\big) \geq \tau(a) - \eta$$

and

$$\tau(b) \geq \tau\big((b'-\epsilon)_+\big) \geq \tau\big((b-2\epsilon)_+\big) \geq \tau(b) - \eta.$$

This yields

$$\big|\tau\big((a'-\epsilon)_+\big) + \tau\big((b'-\epsilon)_+\big) - \big(\tau(a)+\tau(b)\big)\big| < \eta,$$

hence

$$|\tau(a+b) - \tau(a) + \tau(b)| < 2\eta.$$

Since $\eta > 0$ was arbitrary, we obtain

$$\tau(a+b) = \tau(a) + \tau(b),$$

at least if $\tau(a)$, $\tau(b)$, $d_\tau(a)$ and $d_\tau(b)$ are all finite.

If we only assume $\tau(a)$ and $\tau(b)$ to be finite, then for any $\eta > 0$, $d_\tau\big((a-\eta)_+\big)$ and $d_\tau\big((b-\eta)_+\big)$ will be finite, and by the preceding argument we have

$$\tau\big((a-\eta)_+ + (b-\eta)_+\big) = \tau\big((a-\eta)_+\big) + \tau\big((b-\eta)_+\big);$$

by lower semicontinuity this yields

$$\begin{aligned}
\tau(a+b) &= \lim_{\eta\to 0} \tau\big((a-\eta)_+ + (b-\eta)_+\big)\\
&= \lim_{\eta\to 0} \tau\big((a-\eta)_+\big) + \lim_{\eta\to 0} \tau\big((b-\eta)_+\big) = \tau(a) + \tau(b).
\end{aligned}$$

If $\tau(a)$ and $\tau(b)$ are not both finite, there is nothing to show, so we are done. $\square$

ACKNOWLEDGEMENTS. The first author thanks the second, and the other members of Nottingham's math department, for the hospitality during the visit that resulted in this paper. He also thanks the University of Hawaii for their hospitality during the sabbatical year when this work was completed.

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Department of Mathematics, Penn State University, State College, PA 16802, USA
e-mail: nbrown@math.psu.edu

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK
e-mail: wilhelm.winter@nottingham.ac.uk