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On coarse spectral geometry in even dimension

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Abstract. Let σ be the involution of the Roe algebra $C^*|\mathbb{R}|$ which is induced from the reflection $\mathbb{R} \to \mathbb{R};\ x \mapsto -x$. A graded Fredholm module over a separable C^*-algebra A gives rise to a homomorphism $\tilde{\rho}\colon A \to C^*|\mathbb{R}|^\sigma$ to the fixed-point subalgebra. We use this observation to give an even-dimensional analogue of a result of Roe. Namely, we show that the K-theory of this symmetric Roe algebra is $K_0(C^*|\mathbb{R}|^\sigma) \cong \mathbb{Z}$, $K_1(C^*|\mathbb{R}|^\sigma) = 0$, and that the induced map $\tilde{\rho}_*\colon K_0(A) \to \mathbb{Z}$ on K-theory gives the index pairing of K-homology with K-theory.

Résumé. Soit σ l'involution de l'algèbre de Roe $C^*|\mathbb{R}|$ induite par la réflexion $\mathbb{R} \to \mathbb{R};\ x \mapsto -x$. Un module de Fredholm gradué sur une C^*-algèbre séparable A donne lieu à un homomorphisme $\tilde{\rho}\colon A \to C^*|\mathbb{R}|^\sigma$ à valeurs dans la sous-algèbre des éléments invariants. En utilisant cette observation, nous montrons un analogue en dimension paire d'un résultat de Roe. Plus précisément, nous montrons que la K-théorie de cette algèbre de Roe symétrique est $K_0(C^*|\mathbb{R}|^\sigma) \cong \mathbb{Z}$, $K_1(C^*|\mathbb{R}|^\sigma) = 0$ et que l'application induite $\tilde{\rho}_*\colon K_0(A) \to \mathbb{Z}$ coïncide avec l'accouplement entre K-homologie et K-théorie.

1. **Introduction.** In [Roe97], John Roe observed that a Dirac operator D on an odd-dimensional closed manifold M gives rise to a C^*-algebra homomorphism

$$\tilde{\rho}\colon C(M) \to C^*|\mathbb{R}| \tag{1.1}$$

from the continuous functions on M to the Roe algebra of the real line $\mathbb{R}$. The space $\mathbb{R}$ appears because, up to coarse equivalence, it is the spectrum of the self-adjoint operator D. The K-theory of $C^*|\mathbb{R}|$ is

$$K_n(C^*|\mathbb{R}|) \cong \begin{cases} 0, & n = 0, \\ \mathbb{Z}, & n = 1, \end{cases}$$

and the map

$$\tilde{\rho}_*\colon K_1\big(C(M)\big) \to K_1(C^*|\mathbb{R}|) \cong \mathbb{Z} \tag{1.2}$$

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agrees with the index pairing of K-theory with the K-homology class $[D] \in K_1(M)$.

This point of view was extensively developed by Luu [Luu05], who showed that analytic K-homology can be reformulated entirely in the language of coarse spectral geometry. Specifically, let A be a separable C^*-algebra. Luu defined groups $KC^n(A, \mathbb{C})$ whose cycles are $*$-homomorphisms $\rho\colon A \to C^*|\mathbb{R}^n|$,[1] and then proved that $KC^n(A, \mathbb{C}) \cong KK^n(A, \mathbb{C})$. In fact, Luu worked with an arbitrary (σ-unital) coefficient algebra B, to produce groups $KC^n(A, B)$ isomorphic to $KK^n(A, B)$. We choose not to work in that generality here.

Luu's picture of K-homology is aesthetically very pleasing. The price of this elegance, however, is some computational complexity in even dimensions. The isomorphism of KK and KC in even dimension is achieved via a map $KK^0(A, \mathbb{C}) \to KC^2(A, \mathbb{C})$ which requires as input a *balanced* Fredholm module, *i.e.*, a graded Fredholm module of the form $\big(H = H_0 \oplus H_0, \rho = \rho_0 \oplus \rho_0, F = \left(\begin{smallmatrix} 0 & U^* \\ U & 0 \end{smallmatrix}\right)\big)$ for some Hilbert space H_0, representation ρ_0 and Fredholm operator $U\colon H_0 \to H_0$. While every K^0-class can be represented by a balanced Fredholm module, the process of "balancing" is quite heavy-handed. For instance, given a Dirac operator on an even dimensional manifold, the Hilbert space of the associated balanced Fredholm module is an infinite direct sum of L^2-sections of the spinor bundle. (See [HR00, Proposition 8.3.12].) The relationship between the spectrum of U and that of the original operator D is not clear.

In this paper, we describe an alternative approach to controlled spectral geometry in even dimension which is more convenient for geometric applications. Let $s\colon \mathbb{R} \to \mathbb{R}$ denote the reflection through the origin. This induces a $*$-involution σ of the Roe algebra $C^*|\mathbb{R}|$ (see Section 3). Given a graded Fredholm module (H, ρ, D) for A, Roe's construction in fact produces a $*$-homomorphism $\tilde{\rho}\colon A \to C^*|\mathbb{R}|^\sigma$ into the fixed-point algebra of σ. Our main result is the following.

THEOREM 1.1. *The K-theory of the symmetric Roe algebra is*

$$K_n(C^*|\mathbb{R}|^\sigma) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n = 1, \end{cases}$$

and the induced map

$$\tilde{\rho}_*\colon K_0(A) \to K_0(C^*|\mathbb{R}|^\sigma) \cong \mathbb{Z} \tag{1.3}$$

agrees with the index pairing of $[(H, \rho, D)] \in K^0(A)$ with K-theory.

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[1] The most natural coarse structure on $\mathbb{R}^n$ here is the topologically controlled coarse structure associated to the compactification of $\mathbb{R}^n$ by a sphere at infinity. (See [Roe03] for the definition.) If A is separable, it turns out to be equivalent to use the standard metric coarse structure on $\mathbb{R}^n$, although the construction becomes somewhat more technical. The K-theory of $C^*|\mathbb{R}^n|$ is the same in either case.

Following Luu [Luu05], one could then use homomorphisms $\rho\colon A \to C^*|\mathbb{R}|^\sigma$ to *define* even-dimensional cycles in K-homology. We do not pursue this here.

To conclude the introduction, we outline the even-dimensional analogue of the geometric example of [Roe97].

EXAMPLE 1.2. Let M be a compact even-dimensional manifold, and D be a Dirac operator acting on the sections of a graded vector bundle E over M. Functional calculus on D defines a map

$$m\colon C_0(\mathbb{R}) \to \mathcal{B}\big(L^2(E)\big); \quad f \mapsto f(D)$$

which gives $H = L^2(E)$ the structure of a geometric $\mathbb{R}$-Hilbert space (see Section 2), and hence allows us to define a Roe algebra $C^*(|\mathbb{R}|; H)$. The multiplication representation $\rho\colon C(M) \to \mathcal{B}(L^2(E))$ has image in $C^*(|\mathbb{R}|; H)$.

The grading operator γ induces an involution σ of $\mathcal{B}(H)$ by $\sigma(T) = \gamma T \gamma$. Since $\sigma(D) = -D$, we have $\sigma\big(m(f)\big) = m(f \circ s)$, so that σ is the involution of $C^*(|\mathbb{R}|; H)$ induced by the symmetry s. Since $\rho\big(C(M)\big)$ commutes with γ, ρ maps to the fixed-point subalgebra $C^*(|\mathbb{R}|; H)^\sigma$. The induced map on K-theory (1.3) recovers the pairing of $K^0(M)$ with the K-homology class $[D]$.

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2. Preliminaries: the Roe algebra $C^*|\mathbb{R}|$.

We shall use $|\mathbb{R}|$ to denote the real line equipped with the topological coarse structure induced from the two-point compactification $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Thus, a set $E \subseteq \mathbb{R} \times \mathbb{R}$ is *controlled* if for any sequence $(x_n, y_n) \in E$, $x_n \to \infty$ (resp. $-\infty$) if and only if $y_n \to \infty$ (resp. $-\infty$).

We shall refer to a Hilbert space H equipped with a nondegenerate representation $m\colon C_0(\mathbb{R}) \to \mathcal{B}(H)$ as a *geometric* $\mathbb{R}$*-Hilbert space*. By the spectral theorem, m extends naturally to the algebra of Borel functions $B(\mathbb{R})$. We shall typically suppress mention of m in the notation. We use χ_Y to denote the characteristic function of a subset $Y \subset \mathbb{R}$.

An operator $T \in \mathcal{B}(H)$ is *locally compact* if $fT, Tf \in \mathcal{K}(H)$ for all $f \in C_0(\mathbb{R})$. It is *controlled* (for the above topological coarse structure) if for all $R \in \mathbb{R}$ there exists $S \in \mathbb{R}$ such that

$$\chi_{(-\infty,R]} T \chi_{[S,\infty)} = 0, \quad \chi_{[S,\infty)} T \chi_{(-\infty,R]} = 0,$$
$$\chi_{(-R,\infty]} T \chi_{(-\infty,-S]} = 0, \quad \chi_{(-\infty,-S]} T \chi_{[-R,\infty)} = 0.$$

One defines $C^*(|\mathbb{R}|; H)$ as the norm-closure of the locally compact and controlled operators on H. This C^*-algebra is independent of the choice of H as long as H is *ample*, *i.e.*, $m(f)$ is noncompact for all nonzero $f \in C_0(\mathbb{R})$. In that case, the algebra is referred to as the *Roe algebra* $C^*|\mathbb{R}|$.

The following standard facts are easy consequences of the definitions, or alternatively of the coarsely excisive decomposition $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$ (see [HRY93], [HPR97]).

LEMMA 2.1. *Let $T \in C^*(|\mathbb{R}|; H)$. For any $R_1, R_2 \in \mathbb{R}$,*

(i) $\chi_{(-\infty,R_1]}T\chi_{[R_2,\infty)}$ *and* $\chi_{[R_2,\infty)}T\chi_{(-\infty,R_1]}$ *are compact operators.*

(ii) $[T, \chi_{(-\infty,R_1]}]$ *and* $[T, \chi_{[R_2,\infty)}]$ *are compact operators.*

3. Graded Fredholm modules and the symmetric Roe algebra. In what follows, we shall use the unbounded ('Baaj-Julg') picture of K-homology [BJ83]. This is a purely aesthetic choice—see Remark 3.3 for the construction using bounded Fredholm modules.

Let A be a C^*-algebra, and let (H, ρ, D) be a graded unbounded Fredholm module for A, *i.e.*, H is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space, ρ is a representation of A by even operators on H, and D is an odd self-adjoint unbounded operator on H such that

(1) for all $a \in A$, $(1 + D^2)^{-\frac{1}{2}}\rho(a)$ extends to a compact operator,

(2) for a dense set of $a \in A$, $[D, \rho(a)]$ is densely defined and extends to a bounded operator.

Let $\gamma_{\mathrm{ev}}, \gamma_{\mathrm{od}}$ denote the projections onto the even and odd components of H, and $\gamma = \gamma_{\mathrm{ev}} - \gamma_{\mathrm{od}}$ be the grading operator. Let σ be the involution of $\mathcal{B}(H)$ defined by $\sigma\colon T \mapsto \gamma T \gamma$.

Functional calculus on the operator D provides H with a geometric $\mathbb{R}$ structure, namely $m\colon B(\mathbb{R}) \to \mathcal{B}(H)$; $f \mapsto f(D)$. For any $f \in C_0(\mathbb{R})$,

$$\sigma\big(m(f)\big) = f(\gamma.D.\gamma) = f(-D) = m(f \circ s),$$

where $s\colon \mathbb{R} \to \mathbb{R}$ is the reflection in the origin. In coarse language, γ is a covering isometry for s. It follows that σ restricts to an involution of $C^*(|\mathbb{R}|; H)$. The subalgebra fixed by σ will be denoted $C^*(|\mathbb{R}|; H)^\sigma$.

Taking this symmetry into account gives an immediate strengthening of Roe's construction for ungraded Fredholm modules.

PROPOSITION 3.1. *The image of ρ lies in $C^*(|\mathbb{R}|; H)^\sigma$.*

PROOF. The function $f(x) = x(1+x^2)^{-\frac{1}{2}}$ generates $C(\overline{\mathbb{R}})$, and the ideal generated by $g(x) = (1+x^2)^{-\frac{1}{2}}$ is $C_0(\mathbb{R})$. Using [HR00, Theorem 6.5.1], Properties (1) and (2) above imply that $\rho(a) \in C^*(|\mathbb{R}|; H)$ for any $a \in A$. Since $\rho(a)$ is even, $\sigma(a) = \gamma\rho(a)\gamma = \rho(a)$. □

This geometric $\mathbb{R}$-Hilbert space H is not typically ample. However, one can always embed H into an ample geometric $\mathbb{R}$-Hilbert space. For specificity, let us put $\mathcal{H} := H \oplus L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ has its natural geometric $\mathbb{R}$-structure. Extension of operators by zero gives an inclusion $\iota\colon C^*(|\mathbb{R}|; H) \hookrightarrow C^*|\mathbb{R}|$. Put $\tilde{\rho} = \iota \circ \rho\colon A \to C^*|\mathbb{R}|$.

The symmetry $g \mapsto g \circ s$ defines a grading operator on $L^2(\mathbb{R})$. We shall reuse γ to denote the total grading operator on $\mathcal{H}$. Likewise, we use σ to denote conjugation by γ in $\mathcal{B}(\mathcal{H})$. Then $\tilde{\rho}$ has image in $C^*|\mathbb{R}|^\sigma$.

Remark 3.2. In the above, we have employed a specific choice of symmetry $\sigma \in \mathrm{Aut}(C^*|\mathbb{R}|)$ associated to the reflection s of $\mathbb{R}$. For the expert concerned about the uniqueness of this definition, we supply some brief comments without proof. They shall not be needed in what follows.

Let $\mathcal{H}$ be any ample geometric $\mathbb{R}$-Hilbert space. By [Luu05, Prop. 2.2.11(iii)] (following [HRY93]), there exists a unitary $\gamma\colon \mathcal{H} \to \mathcal{H}$ which covers s, in the sense that $(1 \times s)\big(\mathrm{Supp}(\gamma)\big) \subseteq \mathbb{R} \times \mathbb{R}$ is a controlled set. By carrying out the proof of this fact in a way that maintains the reflective symmetry, one can ensure that γ is involutive, $\gamma^2 = 1$. Then $\sigma\colon T \to \gamma T \gamma$ is an involution of $C^*|\mathbb{R}|$. If γ' is another involutive covering isometry for s, then there is a controlled unitary $V \in \mathcal{B}(\mathcal{H})$ such that $\gamma' = V\gamma$ ([Luu05, Prop. 2.2.11(iv)] following [HRY93]). If σ' is conjugation by γ', then $C^*|\mathbb{R}|^{\sigma'} = VC^*|\mathbb{R}|^{\sigma}V^*$. Thus the symmetric Roe algebra $C^*|\mathbb{R}|^{\sigma}$ is unique up to controlled unitary equivalence.

Remark 3.3. The bounded Fredholm module corresponding to (H, ρ, D) is $(H, \rho, F := D(1 + D^2)^{-\frac{1}{2}})$. The map $\phi\colon x \mapsto x(1 + x^2)^{-\frac{1}{2}}$ defines a coarse equivalence from $|\mathbb{R}|$ to the interval $|(-1, 1)|$, with topological coarse structure associated to its two-point compactification $[-1, 1]$. Thus, the bounded picture of K-homology provides a morphism $\rho\colon A \to C^*|(1, -1)| \cong C^*|\mathbb{R}|$.

4. K-theory of the symmetric Roe algebra $C^*|\mathbb{R}|^{\sigma}$.

We continue with the notation above. In particular, we use γ_{ev} and γ_{od} to denote the projections onto the even and odd components of the graded ample geometric $\mathbf{R}$-module $\mathcal{H}$, and we put $\mathcal{H}_{\mathrm{ev}} := \gamma_{\mathrm{ev}}\mathcal{H}$, $\mathcal{H}_{\mathrm{od}} := \gamma_{\mathrm{od}}\mathcal{H}$.

PROPOSITION 4.1. *The K-theory of $C^*|\mathbb{R}|^{\sigma}$ is*

$$K_\bullet(C^*|\mathbb{R}|^{\sigma}) \cong \begin{cases} \mathbb{Z}, & \bullet = 0, \\ 0, & \bullet = 1. \end{cases}$$

Moreover, $K_0(C^*|\mathbb{R}|)^{\sigma}$ is generated by finite rank projections $p \in M_n(C^*|\mathbb{R}|^{\sigma})$, and for such projections, the map to $\mathbb{Z}$ is given by

$$[p] \mapsto \dim p\mathcal{H}_{\mathrm{ev}} - \dim p\mathcal{H}_{\mathrm{od}}.$$

We use a Mayer–Vietoris type argument (*cf.* [HRY93]). Put $Y_+ := [1, \infty)$, $Y_- := (-\infty, -1]$, with their coarse structures inherited from $|\mathbb{R}|$. We will abbreviate $\chi_{Y_\pm}$ as $\chi_\pm$. Since $\mathcal{H}_+ := \chi_+\mathcal{H}$ is an ample geometric Y_+-Hilbert space, we can define the Roe algebra $C^*|Y_+|$ as the corner algebra $C^*(|Y_+|; \mathcal{H}_+) = \chi_+ C^*|\mathbb{R}|\chi_+$. Likewise for $C^*|Y_-|$.

Note that $\sigma(\chi_\pm) = \chi_\mp$, so that σ interchanges $C^*|Y_+|$ and $C^*|Y_-|$. Since $\chi_+\chi_- = 0$, the symmetrization map $(I + \sigma)\colon T \mapsto T + \sigma(T)$ is a $*$-homomorphism from $C^*|Y_+|$ into $C^*|\mathbb{R}|^{\sigma}$. We obtain a morphism of short-exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}_+) & \longrightarrow & C^*|Y_+| & \longrightarrow & C^*|Y_+|/\mathcal{K}(\mathcal{H}_+) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (I+\sigma)} & & \downarrow{\scriptstyle (I+\sigma)} & & \downarrow{\scriptstyle (I+\sigma)} & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H})^{\sigma} & \longrightarrow & C^*|\mathbb{R}|^{\sigma} & \longrightarrow & C^*|\mathbb{R}|^{\sigma}/\mathcal{K}(\mathcal{H})^{\sigma} & \longrightarrow & 0. \end{array} \tag{4.1}$$

LEMMA 4.2. *The right-hand map $(I+\sigma)\colon C^*|Y_+|/\mathcal{K}(\mathcal{H}_+) \to C^*|\mathbb{R}|^\sigma/\mathcal{K}(\mathcal{H})^\sigma$ is an isomorphism.*

PROOF. Let $\psi\colon C^*|\mathbb{R}|^\sigma \to C^*|Y_+|$ denote the cut-down map $T \mapsto \chi_+ T\chi_+$. By using Lemma 2.1(ii), ψ is a homomorphism modulo compacts, so it descends to a homomorphism $\psi\colon C^*|\mathbb{R}|^\sigma/\mathcal{K}(\mathcal{H})^\sigma \to C^*|Y_+|/\mathcal{K}(\mathcal{H})$. By Lemma 2.1(i), for any $T \in C^*|\mathbb{R}|^\sigma$ we have

$$T \equiv \chi_+ T\chi_+ + \chi_- T\chi_- \mod \mathcal{K}(\mathcal{H})^\sigma,$$

so ψ is inverse to $(I+\sigma)$. □

LEMMA 4.3. *We have $\mathcal{K}(\mathcal{H})^\sigma \cong \mathcal{K}(\mathcal{H}_{\mathrm{ev}}) \oplus \mathcal{K}(\mathcal{H}_{\mathrm{od}})$ via $T \mapsto T\gamma_{\mathrm{ev}} \oplus T\gamma_{\mathrm{od}}$. In particular, $K_0(\mathcal{K}(\mathcal{H})^\sigma) \cong \mathbb{Z}\oplus\mathbb{Z}$ via the map which sends the class of a projection p to $\big(\dim(p\mathcal{H}_{\mathrm{ev}}), \dim(p\mathcal{H}_{\mathrm{od}})\big)$.*

PROOF. Note that any $T \in C^*|\mathbb{R}|^\sigma$ commutes with γ, so $T \mapsto T\gamma_{\mathrm{ev}} \oplus T\gamma_{\mathrm{od}}$ is indeed a homomorphism. The inverse homomorphism is $T_1 \oplus T_2 \mapsto T_1 + T_2$. □

LEMMA 4.4. *Under the identifications $K_0\big(\mathcal{K}(\mathcal{H}_+)\big) \cong \mathbb{Z}$ and $K_0\big(\mathcal{K}(\mathcal{H})^\sigma\big) \cong \mathbb{Z}\oplus\mathbb{Z}$, the map $(I+\sigma)_*$ is $n \mapsto (n,n)$.*

PROOF. Let p be a projection in $\mathcal{K}(\mathcal{H}_+)$. Then $p = \chi_{Y_+} p \chi_{Y_+}$, so $p\gamma = \chi_{Y_+} p\gamma\chi_{Y_-}$, and hence $\mathrm{Tr}(p\gamma) = 0$. Using $\gamma_{\mathrm{ev}\,/\,\mathrm{od}} = \frac{1}{2}(1\pm\gamma)$, this gives $\mathrm{Tr}(p\gamma_{\mathrm{ev}}) = \mathrm{Tr}(p\gamma_{\mathrm{od}}) = \frac{1}{2}\mathrm{Tr}(p)$. Similarly, $\mathrm{Tr}\big(\sigma(p)\gamma_{\mathrm{ev}}\big) = \mathrm{Tr}\big(\sigma(p)\gamma_{\mathrm{od}}\big) = \frac{1}{2}\mathrm{Tr}\big(\sigma(p)\big) = \frac{1}{2}\mathrm{Tr}(p)$. Hence, $\mathrm{Tr}\big((I+\sigma)(p)\gamma_{\mathrm{ev}}\big) = \mathrm{Tr}\big((I+\sigma)(p)\gamma_{\mathrm{od}}\big) = \mathrm{Tr}(p)$, and the result follows from the previous lemma. □

By [Roe96, Proposition 9.4], $C^*|Y_+|$ has trivial K-theory. The boundary maps in K-theory induced from the diagram (4.1) give

$$\begin{array}{ccccc} K_1\big(C^*|Y_+|/\mathcal{K}(\mathcal{H}_+)\big) & \xrightarrow[\cong]{\partial} & K_0\big(\mathcal{K}(\mathcal{H}_+)\big) \cong \mathbb{Z} & & n \\ {\scriptstyle (I+\sigma)_*}\downarrow\cong & & {\scriptstyle (I+\sigma)_*}\downarrow & & \downarrow \\ K_1\big(C^*|\mathbb{R}|^\sigma/\mathcal{K}(\mathcal{H})^\sigma\big) & \xrightarrow{\partial} & K_0\big(\mathcal{K}(\mathcal{H})^\sigma\big) \cong \mathbb{Z}\oplus\mathbb{Z} & & (n,n). \end{array} \tag{4.2}$$

We see that $K_1\big(C^*|\mathbb{R}|^\sigma/\mathcal{K}(\mathcal{H})^\sigma\big) \cong \mathbb{Z}$, and the image of its boundary map into $K_0\big(\mathcal{K}(\mathcal{H})^\sigma\big)$ is $\{(n,n) \mid n \in \mathbb{Z}\}$. The corresponding diagram in the other degree gives $K_0\big(C^*|\mathbb{R}|^\sigma/\mathcal{K}(\mathcal{H})^\sigma\big) \cong 0$.

Now the six-term exact sequence associated to the bottom row of (4.1) becomes

$$\begin{array}{ccccccc} (n,n) & & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & K_0(C^*|\mathbb{R}|^\sigma) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & & & \downarrow \\ n & & \mathbb{Z} & \longleftarrow & K_1(C^*|\mathbb{R}|^\sigma) & \longleftarrow & 0 \end{array}$$

Thus, $K_0(C^*|\mathbb{R}|^\sigma) \cong \mathbb{Z}$ and $K_1(C^*|\mathbb{R}|^\sigma) \cong 0$. With an appropriate choice of sign, top-left horizontal map is given by $(m,n) \mapsto m-n$. Applying Lemma 4.3, this completes the proof of Proposition 4.1.

5. The index pairing.

Let $\theta \in K^0(A)$ be the K-homology class of a graded unbounded Fredholm module (H,ρ,D), and put $F := D(1+D^2)^{-\frac{1}{2}}$. Let p be a projection in $M_n(A)$. The index pairing $K^0(A) \times K_0(A) \to \mathbb{Z}$ is given by

$$(\theta,[p]) := \mathrm{Index}[\rho(p)(F \otimes I_n)\rho(p)\colon \rho(p)H^n_{\mathrm{ev}} \to \rho(p)H^n_{\mathrm{od}}],$$

(where I_n denotes the identity in $M_n(\mathbb{C})$.)

Now apply the construction of Section 3 to θ. Let $P = \tilde{\rho}(p) \in M_n(C^*|\mathbb{R}|^\sigma)$, and let f denote the function $f(x) = x(1+x^2)^{-\frac{1}{2}}$, as represented on the geometric $|\mathbb{R}|$-Hilbert space $\mathcal{H}$. Then

$$(\theta,[p]) = \mathrm{Index}\big(P(f \otimes I_n)P\colon P\mathcal{H}^n_{\mathrm{ev}} \to P\mathcal{H}^n_{\mathrm{od}}\big).$$

The right-hand side here depends only on the class of P in $K_0(C^*|\mathbb{R}|^\sigma)$. By Proposition 4.1, we may therefore replace P by a finite rank projection Q, and the index is

$$\begin{aligned}(\theta,[p]) &= \mathrm{Index}\big(Q(f \otimes I_n)Q : Q\mathcal{H}^n_{\mathrm{ev}} \to Q\mathcal{H}^n_{\mathrm{od}}\big)\\ &= \dim(Q\mathcal{H}^n_{\mathrm{ev}}) - \dim(Q\mathcal{H}^n_{\mathrm{od}})\\ &= [Q] = \tilde{\rho}_*[p].\end{aligned}$$

This completes the proof of Theorem 1.1.

Given the above results, it is natural to expect a reformulation of $KK^0(A,\mathbb{C})$ in the spirit of Luu. Indeed, one can define a group $KC^0_\sigma(A,\mathbb{C})$ as follows. Cycles are morphisms from A into the symmetric Roe algebra $C^*|\mathbb{R}|^\sigma$. Equivalence of cycles is generated by controlled unitary equivalences (preserving the involution γ) and weak homotopies (respecting the symmetry σ). Then $KC^0_\sigma(A,\mathbb{C}) \cong KK^0(A,\mathbb{C})$. We shall not develop this in detail here, as the results follow [Luu05] closely.

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