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On the Structure of the Set of Integral Points Inside a Ball

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Abstract. We prove that for each open ball in $\mathbb{R}^n$ of radius $r \geq \frac{\sqrt{n+3}}{2}$, its center is contained in the convex hull of all integral points inside it. We also show that this estimate is sharp, *i.e.*, for balls of radius $r < \frac{\sqrt{n+3}}{2}$, the property does not hold.

Résumé. Nous prouvons que, pour chaque balle ouverte dans $\mathbb{R}^n$ de radius $r \geq \frac{\sqrt{n+3}}{2}$, le centre se tient dans une enveloppe convexe de points à l'intérieur avec des coordonnées entières. Nous allons également montrer que cette estimation est forte, c'est-à-dire que la propriété ne tient pas pour les balles de radius $r < \frac{\sqrt{n+3}}{2}$.

In a forthcoming paper [KK], Askold Khovanskii and Kiumars Kaveh have used surprisingly deep algebraic consequences of the following simple fact. Suppose we are given an open ball B in $\mathbb{R}^n$ of radius $r > 0$. Then for r big enough one can guarantee that no matter how B is located in the ambient space, its center belongs to the convex hull of all integral points inside B, *i.e.*, points with coordinates $(x_1, \dots, x_n)$ where $x_i \in \mathbb{Z}$ for $1 \leq i \leq n$. This is clearly true if $r > \sqrt{n}$, since in that case B contains the whole cube with integral vertices and edgelength 1, which covers its center. This elementary estimate is sufficient for the purposes of paper [KK], but it gives rise to the following geometric question: in general, for which values of r does the described above property hold? The answer is given by the following theorem.

Theorem 1.
(i) *For any open ball B in $\mathbb{R}^n$ of radius $r \geq \frac{\sqrt{n+3}}{2}$, its center belongs to the convex hull of all integral points inside B.*
(ii) *For each $r < \frac{\sqrt{n+3}}{2}$, there is a way to to choose an open ball B in $\mathbb{R}^n$ of radius r so that its center will not be contained in the convex hull of all integral points inside the ball.*

Proof. We begin with the second part of the theorem. An example is constructed by taking the center of the ball near the center of the facet of elementary lattice cube. More formally, take for the center a point $A = (\frac{1}{2}, \dots, \frac{1}{2}, \epsilon)$ with

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small positive ϵ. The square of distance from it to any integral point with nonzero last coordinate is at least $(n-1)(\frac{1}{2})^2+(1-\epsilon)^2 > r^2$ for sufficiently small ϵ. Consequently, all integral points inside an open ball of radius r with center at A have zero last coordinate, so this also holds for any point in their convex hull. But the last coordinate of A is not $0 \Rightarrow$ it doesn't belong to the mentioned convex hull.

For the proof of (i) let $p = (p_1, \dots, p_n)$ be a center of B. Assume that p doesn't belong to the convex hull K of all integral points in $B \Rightarrow$ there exists a hyperplane ℓ that separates p from K and it is given by the equation $u_1x_1 + \cdots + u_nx_n = u$. After possibly multiplying this equation by -1 we may assume that $u_1p_1 + \cdots + u_np_n > u$. To get a contradiction we must find a point $q = (q_1, \dots, q_n) \in K$ which lies on the same side of ℓ as p. Let c_i be the closest integer to p_i, so that $e_i = |p_i - c_i| \le \frac{1}{2}$. Let $d_i = 1$, if $p_i - c_i \ge 0$ and $d_i = -1$, if $p_i - c_i < 0$. So $p_i - c_i = d_ie_i$. If we denote by I_i the closed interval with endpoints c_i and $c_i + d_i$, then $I = I_1 \times \cdots \times I_n$ is an elementary lattice cube which contains p. A good choice for q would be one of the vertices of I, but the problem is that probably not all of them are contained in B. So we need to choose wisely a vertex $q = (q_1, \dots, q_n)$ of I, so that two inequalities will be met: $(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2 < r^2$ (if q is inside B) and $u_1(q_1 - p_1) + \cdots + u_n(q_n - p_n) \ge 0$ (as a consequence, q is from the same side of ℓ as p). We will require the following lemma.

LEMMA 1. *Suppose $a_1, \dots, a_n, b_1, \dots, b_n$ are real numbers, such that $0 \le a_i \le 1$ for every $1 \le i \le n$. Then there exist integers $z_1, \dots, z_n$, $y_1, \dots, y_n$, such that each z_i is 0 or 1, $z_i + y_i = 1$ for every i and the following inequalities are true:*

$$(*) \qquad z_1a_1 + \cdots + z_na_n < \frac{3 + 2(a_1 + \cdots + a_n) - (a_1^2 + \cdots + a_n^2)}{4}$$

$$(**) \qquad y_1b_1 + \cdots + y_nb_n \le \frac{b_1(1 + a_1) + \cdots + b_n(1 + a_n)}{2}.$$

We will postpone the proof of the lemma a bit and show how to finish the proof of the theorem using it. Let $a_i = 1 - 2e_i$ (so $0 \le a_i \le 1$). Apply lemma for numbers $a_1, \dots, a_n, b_1 = u_1d_1, \dots, b_n = u_nd_n$ to find $z_1, \dots, z_n$, $y_1, \dots, y_n$ as specified above. Now let $q_i = c_i$ if $z_i = 0$ and $q_i = c_i + d_i$ if $z_i = 1$. Take $q = (q_1, \dots, q_n)$. It is an integral point which belongs to B, because the square of distance from it to p is

$$\begin{aligned}
&(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2 \\
&\quad = \big((p_1 - c_1)^2 + \cdots + (p_n - c_n)^2\big) \\
&\qquad + z_1\big(d_1^2 - 2(p_1 - c_1)d_1\big) + \cdots + z_n\big(d_n^2 - 2(p_n - c_n)d_n\big)
\end{aligned}$$

$$= (e_1^2 + \cdots + e_n^2) + (z_1 + \cdots + z_n) - 2(z_1e_1 + \cdots + z_ne_n)$$
$$= (z_1a_1 + \cdots + z_na_n) + \big((\frac{1-a_1}{2})^2 + \cdots + (\frac{1-a_n}{2})^2\big)$$
$$< \frac{n+3}{4} \le r^2$$

by $(*)$. So $q \in K$.

Observe also that

$$u_1(q_1 - p_1) + \cdots + u_n(q_n - p_n)$$
$$= u_1(c_1 - p_1) + \cdots + u_n(c_n - p_n) + (u_1z_1d_1 + \cdots + u_nz_nd_n)$$
$$= b_1(z_1 - e_1) + \cdots + b_n(z_n - e_n)$$
$$= b_1(1 - y_1 - \frac{1-a_1}{2}) + \cdots + b_n(1 - y_n - \frac{1-a_n}{2})$$
$$= \frac{b_1(1+a_1) + \cdots + b_n(1+a_n)}{2} - (b_1y_1 + \cdots + b_ny_n) \ge 0$$

by $(**)$. So we get a contradiction and this finishes the proof of the theorem. □

It remains to prove the lemma.

Proof of Lemma 1. First observe that we may assume that all b_i's are nonnegative. Indeed, suppose we know how to deal with this case, but some of our b_i's are negative. Change all negative numbers to 0 and obtain the new set $\{b_i'\}$. For $a_1, \dots, a_n, b_1', \dots, b_n'$ find $z_1, \dots, z_n$, $y_1, \dots, y_n$ as specified above. We may assume that if $b_i' = 0$, $y_i = 1$ (otherwise change it to 1 and z_i to 0 and inequalities will still be true). Clearly, this choice of y_i's and z_i's works for initial a_i's and b_i's.

Denote the right-hand side of $(*)$ by M and the right-hand side of $(**)$ by N. The main idea of the proof is to begin by replacing the condition $z_i = 0$ or 1 by a milder condition $0 \le z_i \le 1$. For such case one can take $z_i = \frac{2-a_i}{4}$, $y_i = \frac{2+a_i}{4}$. Then $z_1a_1 + \cdots + z_na_n = M - \frac{3}{4} < M$ and $y_1b_1 + \cdots + y_nb_n = N - \frac{a_1b_1+\cdots+a_nb_n}{4} \le N$. Now we start changing coefficients y_i's and z_i's in such a way that $z_1a_1 + \cdots + z_na_n$ and $y_1b_1 + \cdots + y_nb_n$ will not be increasing and conditions $0 \le z_i \le 1$, $y_i = 1 - z_i$ are preserved. And we will be doing that until all but possibly one of z_i's become integers. Suppose at some moment at least two of z_i's are not integers. Without loss of generality assume that these are z_1 and z_2 and $a_1b_2 - a_2b_1 \ge 0$. We may also assume that both b_1 and b_2 are nonzero, because if one of them is zero, we will change the corresponding y_j to 1 and z_j to 0 and increase the number of integers among z_i's. Now start increasing y_1 with speed 1, decreasing z_1 with speed 1, decreasing y_2 with speed $\frac{b_1}{b_2}$, increasing z_2 with speed $\frac{b_1}{b_2}$. During this continuous process $y_1b_1 + \cdots + y_nb_n$ will not be changing; $z_1a_1 + \cdots + z_na_n$ will be decreasing with a nonnegative speed

$$a_1 - \frac{a_2b_1}{b_2} = \frac{a_1b_2 - a_2b_1}{b_2}.$$

Proceed until one of z_1, z_2, y_1, y_2 will become an integer. Since during all process $y_1 + z_1 = 1$, $y_2 + z_2 = 1$, we again increased the number of integers among z_i's. After performing such operations several times we will finally arrive to a situation in which every z_i but possibly one (without loss of generality z_1) is 0 or 1 and nonstrict $(*)$ is true with the right-hand side replaced by $M - \frac{3}{4}$ and $(**)$ is true with the right-hand side replaced by $N - \frac{a_1b_1+\cdots+a_nb_n}{4}$. If $a_1 = 0$ or $z_1 > 1 - \frac{3}{4a_1}$, we can make z_1 equal to 1 and y_1 equal to 0, so that both $(*)$ and $(**)$ will be true. Otherwise, $y_1 \geq \frac{3}{4a_1} \Rightarrow 1 - y_1 \leq \frac{a_1}{4}$, because $1 - \frac{3}{4x} \leq \frac{x}{4}$ for $x \in (0, 1]$. So we can make y_1 equal to 1 and z_1 equal to 0 and $(*)$ and $(**)$ will be true, because $\frac{a_1b_1}{4} \leq \frac{a_1b_1+\cdots+a_nb_n}{4}$. □

Another much simpler question coming from the same algebraic motivation of paper [KK] is this. Suppose we are given an open ball B in $\mathbb{R}^n$ of radius $r > 0$. For which r one can guarantee that no matter how B is located in the ambient space the differences $a - b$ where a and b are integral points in B generate the whole group $\mathbb{Z}^n$? The answer to this problem turns out to be almost the same as for the first. For $r \leq \frac{\sqrt{n+3}}{2}$ consider a ball with center at $(\frac{1}{2}, \ldots, \frac{1}{2}, 0)$ and radius r. Same explanations as provided above show that all integral points inside it have zero last coordinate $\Rightarrow$ their differences also have zero last coordinate and don't generate $\mathbb{Z}^n$. It remains to prove the following.

PROPOSITION 2. *For any open ball B in $\mathbb{R}^n$ of radius $r > \frac{\sqrt{n+3}}{2}$, differences $a - b$ where a and b are integral points in B generate $\mathbb{Z}^n$.*

PROOF. Suppose B has a center at $p = (p_1, \ldots, p_n)$. Let c_i be the closest integer to p_i. Point $(c_1, \ldots, c_n)$ belongs to B, because $(c_1-p_1)^2+\cdots+(c_n-p_n)^2 \leq \frac{n}{4} < r$. Let $d_i = 1$, if $p_i - c_i \geq 0$ and $d_i = -1$, if $p_i - c_i < 0$. Fix i and consider a point $(s_1, \ldots, s_n)$ where $s_j = c_j$ for $j \neq i$ and $s_i = c_i + d_i$. It also is in B, because $|s_i - p_i| \leq 1$, $|s_j - p_j| \leq \frac{1}{2}$ for $j \neq i \Rightarrow (s_1 - p_1)^2 + \cdots + (s_n - p_n)^2 \leq \frac{n-1}{4} + 1 < r$. The difference between $(s_1, \ldots, s_n)$ and $(c_1, \ldots, c_n)$ is $(0, \ldots, d_i, \ldots, 0)$. Clearly such elements (considered for i from 1 to n) generate $\mathbb{Z}^n$. □

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REFERENCES

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