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Banach-valued Holomorphic Functions on the Maximal Ideal Space of H^∞

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Abstract. We study Banach-valued holomorphic functions defined on open subsets of the maximal ideal space of the Banach algebra H^∞ of bounded holomorphic functions on the unit disk $\mathbf{D} \subset \mathbf{C}$ with pointwise multiplication and supremum norm. In particular, we establish the vanishing of the cohomology of sheaves of germs of such functions and, solving a Banach-valued corona problem for H^∞, prove that the maximal ideal space of the algebra $H^\infty_{\mathrm{comp}}(A)$ of holomorphic functions on $\mathbf{D}$ with relatively compact images in a commutative unital complex Banach algebra A is homeomorphic to the direct product of maximal ideal spaces of H^∞ and A. All proofs are presented in `arXiv:1103.2347v1`.

Résumé. Nous étudions des fonctions holomorphes à valeurs dans un espace de Banach et définies sur des sous ensembles ouverts de l'espace idéal maximal de l'algèbre de Banach H^∞ des fonctions holomorphes bornées sur le disque unité $\mathbf{D} \subset \mathbf{C}$ munies de la multiplication ponctuelle et de la norme du supremum. En particulier, nous établissons que la cohomologie des faisceaux des germes de ces fonctions est nulle et, par le biais de la résolution d'un problème de type corona pour H^∞ à valeurs dans un espace de Banach, montrons que l'espace idéal maximal de l'algèbre $H^\infty_{\mathrm{comp}}(A)$ des fonctions holomorphes sur $\mathbf{D}$ et à image relativement compacte dans une algbre complexe commutative unitale de Banach A est homomorphe au produit direct des espaces idéaux maximaux de H^∞ et A. Toutes les preuves sont présentées dans `arXiv:103.2347v1`.

1. Formulation of main results.

1.1. In this paper we propose a new approach to the study of the Banach algebra H^∞ of bounded holomorphic functions on the unit disk $\mathbf{D} \subset \mathbf{C}$ with pointwise multiplication and supremum norm. It is based on a new method for solving Banach-valued $\bar\partial$-equations on $\mathbf{D}$ that allows one to develop a Stein-like theory for Banach-valued holomorphic functions defined on open subsets of the maximal ideal space of H^∞. Specifically, as in the classical theory of Stein spaces, we establish the vanishing of the cohomology of sheaves of germs of such

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functions, solve the second Cousin problem and prove Runge-type approximation theorems for them. We then apply the developed technique to the study of the algebra $H^\infty_{\rm comp}(A)$ of holomorphic functions on $\mathbf{D}$ with relatively compact images in a commutative unital complex Banach algebra A. This class of algebras includes, e.g., the slice algebras for H^∞. In particular, solving a Banach-valued corona problem for H^∞, we prove that the maximal ideal space of $H^\infty_{\rm comp}(A)$ is homeomorphic to the direct product of maximal ideal spaces of H^∞ and A. This result would also follow from Theorem 1.1 below if we knew that H^∞ has the Grothendieck approximation property (which is still an open problem).

Recall that for a commutative unital complex Banach algebra A with dual space A^* the *maximal ideal space* $M(A)$ of A is the set of nonzero homomorphisms $A \to \mathbf{C}$ equipped with the *Gelfand topology*, the weak* topology induced by A^*. It is a compact Hausdorff space contained in the unit ball of A^*. Let $C\big(M(A)\big)$ be the algebra of continuous complex-valued functions on $M(A)$ with supremum norm. The Gelfand transform $\hat{\ }: A \to C\big(M(A)\big)$, defined by $\hat{a}(\varphi) := \varphi(a)$, is a nonincreasing-norm morphism of algebras that allows to thought of elements of A as continuous functions on $M(A)$. If the Gelfand transform is an isometry (as for H^∞), then A is called a *uniform algebra.*

Throughout the paper all Banach algebras are assumed to be complex, commutative and unital.

In the case of H^∞ evaluation at a point of $\mathbf{D}$ is an element of $M(H^\infty)$, so $\mathbf{D}$ is naturally embedded into $M(H^\infty)$ as an open subset. The famous Carleson corona theorem [C1] asserts that $\mathbf{D}$ is dense in $M(H^\infty)$.

Next, given Banach algebras $A \subset C(X)$, $B \subset C(Y)$ (X and Y are compact Hausdorff spaces) their *slice algebra* is defined as

$$S(A;B) := \{f \in C(X\times Y); f(\,\cdot\,;y) \in A \text{ for all } y \in Y; f(x;\,\cdot\,) \in B \text{ for all } x \in X\}.$$

The main problem concerning algebra $S(A;B)$ is to determine whether it coincides with $A \otimes B$, the closure in $C(X \times Y)$ of the symmetric tensor product of A and B. For instance, this is true if either A or B have the *approximation property.* The latter is an immediate consequence of the following result of Grothendieck [G, Section 5.1].

Let $A \subset C(X)$ be a closed subspace, B be a complex Banach space and $A_B \subset C(X;B)$ be the Banach space of all continuous B-valued functions f such that $\varphi(f) \in A$ for any $\varphi \in B^*$. By $A \otimes B$ we denote completion of symmetric tensor product of A and B with respect to norm

$$(1.1)\qquad \Big\|\sum_{k=1}^{m} a_k \otimes b_k\Big\| := \sup_{x\in X}\Big\|\sum_{k=1}^{m} a_k(x)b_k\Big\|_B \quad \text{with } a_k \in A,\ b_k \in B.$$

Theorem 1.1 (Grothendieck). *The following statements are equivalent:*
(1) *A has the approximation property;*
(2) *$A \otimes B = A_B$ for every Banach space B.*

Recall that a Banach space B is said to have the approximation property if, for every compact set $K \subset B$ and every $\varepsilon > 0$, there exists an operator $T\colon B \to B$ of finite rank so that $\|Tx - x\|_B \le \varepsilon$ for every $x \in K$.

Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space H^∞. (The strongest result in this direction due to Bourgain and Reinov [BR, Theorem 9] states that H^∞ has the approximation property "up to logarithm".) The first example of a space which fails to have the approximation property was constructed by Enflo [E]. Since Enflo's work several other examples of such spaces were constructed; for the references see, *e.g.*, [L].

In the paper, we show that H^∞ has the approximation property if and only if it has this property in some open neighbourhoods of trivial Gleason parts of $M(H^\infty)$.

1.2. Let $U \subset M(H^\infty)$ be an open subset and B be a complex Banach space.

DEFINITION 1.2. A continuous function $f \in C(U;B)$ is said to be B-valued holomorphic if its restriction to $U \cap \mathbf{D}$ is B-valued holomorphic in the usual sense.

By $\mathcal{O}(U;B)$ we denote the vector space of B-valued holomorphic functions on U.

If $f \in \mathcal{O}(U;B)$, then for every open $V \Subset U$, the restriction $f|_{V\cap\mathbf{D}}$ belongs to the Banach space $H^\infty_{\rm comp}(V \cap \mathbf{D}; B)$ of B-valued holomorphic functions g on $V \cap \mathbf{D}$ with relatively compact images and with norm $\|g\| := \sup_{z\in V\cap\mathbf{D}} \|g(z)\|_B$. Conversely, we have the following proposition.

PROPOSITION 1.3. *Let $f \in \mathcal{O}(U\cap\mathbf{D};B)$ satisfy $f|_{V\cap\mathbf{D}} \in H^\infty_{\rm comp}(V\cap\mathbf{D};B)$ for each open $V \Subset U$. Then there exists a unique function $\tilde f \in \mathcal{O}(U;B)$ such that $\tilde f|_{U\cap\mathbf{D}} = f$.*

By $\mathcal{O}^B_{M(H^\infty)}$ we denote the sheaf of germs of B-valued holomorphic functions on $M(H^\infty)$.

THEOREM 1.4. $H^k\big(M(H^\infty); \mathcal{O}^B_{M(H^\infty)}\big) = 0.$

Here $H^k(X;\mathcal{J})$ stands for the k-th *Čech cohomology group* of a sheaf of abelian groups $\mathcal{J}$ defined on a Hausdorff topological space X.

The proof of Theorem 1.4 is based on a new method for solving certain Banach-valued $\bar\partial$-equations on $\mathbf{D}$.

A function h on an open subset $U \subset M(H^\infty)$ is said to be *meromorphic* if $h = \frac{f}{g}$, where $f, g \in \mathcal{O}(U)$ $(:= \mathcal{O}(U;\mathbf{C}))$ and g is not identically zero. The set of meromorphic functions on U is denoted by $\mathcal{M}(U)$. For $U = M(H^\infty)$ the class $\mathcal{M}(U)$ consists of functions $h = \frac{f}{g}$ with $f, g \in H^\infty$. (Here and below H^∞ is defined on $M(H^\infty)$ by means of the Gelfand transform.)

A (Cartier) *divisor* on $M(H^\infty)$ consists of pairs (U_i, h_i), where (U_i) is an open cover of $M(H^\infty)$ and $h_i \in \mathcal{M}(U_i)$, such that for all $U_i \cap U_j \neq \varnothing$ functions $\frac{h_i}{h_j} \in \mathcal{O}(U_i \cap U_j)$ and are nowhere zero. As a corollary of Theorem 1.4, we obtain the solution of the second Cousin problem on $M(H^\infty)$.

Theorem 1.5. *For any divisor $D = \{(U_i, h_i)\}_{i\in I}$ on $M(H^\infty)$ there exist a meromorphic function $h_D \in \mathcal{M}\big(M(H^\infty)\big)$ and a family of nowhere vanishing functions $c_i \in \mathcal{O}(U_i)$, $i \in I$, such that*

$$h_D|_{U_i} = h_i \cdot c_i \quad \text{for all } i \in I.$$

Remark 1.6. The statement is equivalent to the fact that any holomorphic line bundle on $M(H^\infty)$ (*i.e.*, a line bundle determined on an open cover of $M(H^\infty)$ by a holomorphic cocycle) is holomorphically trivial. The general problem of triviality of a holomorphic Banach vector bundle on $M(H^\infty)$ is considered in `arXiv:1103.5237`. In particular, it is shown that any finite rank holomorphic vector bundle on $M(H^\infty)$ is holomorphically trivial.

Another approach to meromorphic functions on $M(H^\infty)$ was proposed in [BG].

Our next result is a Runge-type approximation theorem for Banach-valued holomorphic functions defined on subsets of $M(H^\infty)$.

A compact subset $K \subset M(H^\infty)$ is called *holomorphically convex* if for any $x \notin K$ there is $f \in H^\infty$ such that $\max_K |f| < |f(x)|$.

Theorem 1.7. *Any B-valued holomorphic function defined on a neigbourhood of a holomorphically compact set $K \subset M(H^\infty)$ can be uniformly approximated on K by functions from $\mathcal{O}\big(M(H^\infty); B\big)$.*

In fact it suffices to prove the result for K a *polyhedron*, *i.e.*, a set of the form $\Pi(F_\ell) := \{x \in M(H^\infty); \max_{1\le j\le \ell} |f_j(x)| \le 1,\ F_\ell := (f_1, \dots, f_\ell) \in (H^\infty)^\ell\}$. Let $\dot{\Pi}(F_\ell)$ denote intersection of the interior of $\Pi(F_\ell)$ with $\mathbf{D}$. Then, in view of Proposition 1.3, we can reformulate Theorem 1.7 as follows: *Any function in $H^\infty_{\mathrm{comp}}\big(\dot{\Pi}(F_\ell); B\big)$ can be uniformly approximated on each $\Pi(r \cdot F_\ell)$, $r > 1$, by functions from $H^\infty_{\mathrm{comp}}(B) := H^\infty_{\mathrm{comp}}(\mathbf{D}; B)$.*

Remark 1.8. With regard to Theorem 1.7 one may ask about an analog of the Mergelyan theorem. For a polyhedron the question can be stated as follows.

> Is it true that any function in $C\big(\Pi(F_\ell); B\big)$ holomorphic on $\dot{\Pi}(F_\ell)$ can be uniformly approximated on $\Pi(F_\ell) \cap \mathbf{D}$ by functions from $H^\infty_{\mathrm{comp}}(B)$?

1.3. In this part we study Banach algebras $H^\infty_{\mathrm{comp}}(A)$ of holomorphic functions on $\mathbf{D}$ with relatively compact images in a commutative complex unital Banach algebra A. This class of algebras includes, *e.g.*, slice algebras $S(H^\infty; \cdot)$. Note that the restriction map to $\mathbf{D}$ induces an isomorphism between $\mathcal{O}\big(M(H^\infty); A\big)$ and $H^\infty_{\mathrm{comp}}(A)$, *cf.* Proposition 1.3.

Theorem 1.9. *Let* $f_1, \dots, f_m$, $f \in \mathcal{O}\big(M(H^\infty); A\big)$. *Then* f *belongs to the ideal* $I \subset \mathcal{O}\big(M(H^\infty); A\big)$ *generated by* $f_1, \dots, f_m$ *if and only if there exists a finite open cover* $(U_k)_{1 \le k \le \ell}$ *of* $M(H^\infty)$ *such that, for every* $1 \le k \le \ell$, *the function* $f|_{U_k}$ *belongs to the ideal* $I_k \subset \mathcal{O}(U_k; A)$ *generated by functions* $f_1|_{U_k}, \dots, f_m|_{U_k}$.

In the proof we use a standard argument involving Koszul complexes which reduces the statement to a question on existence of bounded on the boundary solutions of certain A-valued $\bar\partial$-equations on $\mathbf{D}$ similar to those in Wolff's proof of Carleson's corona theorem, see, *e.g.*, [Ga, Chapter VIII.2]. However, since the target space A may be infinite dimensional, the classical duality method allowing to get such solutions for scalar $\bar\partial$-equations does not work anymore. We use instead some topological properties of $M(H^\infty)$ together with a theorem which provides bounded solutions of A-valued $\bar\partial$-equations with 'supports' in the set of nontrivial Gleason parts of $M(H^\infty)$. The fact that $\mathbf{D}$ is dense in $M(H^\infty)$ is not used in the proof; hence, Theorem 1.9 gives yet another proof of the corona theorem for H^∞.

As a corollary we obtain the following theorem.

Theorem 1.10. $M\big(H^\infty_{\text{comp}}(A)\big) \cong M(H^\infty) \times M(A)$.

This would follow from Theorem 1.1 if we knew that H^∞ has the approximation property. However, Theorem 1.9 yields also results not following from this property; one of them, an analog of Wolff's theorem, see, *e.g.*, [Ga, Chapter VIII, Theorem 2.3], is presented below.

Recall that $M(H^\infty)$ is disjoint union of an open subset M_a of nontrivial Gleason parts (*analytic disks*) and a closed subset M_s of trivial (one-pointed) Gleason parts.

Theorem 1.11. *Let the algebra* $A \subset C(X)$, X *a compact Hausdorff space, be self-adjoint with respect to the complex conjugation. Fix an* $\omega \in C\big([0,\infty)\big)$ *positive on* $(0,\infty)$ *such that* $\lim\limits_{t \to 0^+} \frac{\omega(t)}{t^3} = 0$. *Assume that* $f_1, \dots, f_m$, $f \in S(A) := S(H^\infty; A)$ *satisfy*

(a)

$$|f(z)| \le c \cdot \omega\big(\max_{1 \le j \le m} |f_j(z)|\big) \quad \textit{for some } c > 0 \textit{ and all } z \in \mathbf{D} \times X;$$

(b)

$$\max_{1 \le j \le m} |f(x)| \ge \delta \quad \textit{for some } \delta > 0 \textit{ and all } x \in M_s \times X.$$

Then f *belongs to the ideal* $I \subset S(A)$ *generated by* $f_1, \dots, f_m$.

Example 1.12. A sequence $\{z_n\} \subset \mathbf{D}$ is called *interpolating* for H^∞ if the interpolation problem $f(z_n) = w_n$, $n \in \mathbf{N}$, has a solution $f \in H^\infty$ for every bounded sequence of complex numbers $\{w_n\}$. A *Blaschke product* $B(z) := \prod_{j \ge 1} \frac{z - z_j}{1 - \bar z_j z}$, $z \in \mathbf{D}$, is called interpolating if its set of zeros $\{z_j\}$ is an interpolating sequence for H^∞.

Consider interpolating Blaschke products $B_1,\dots,B_m$. Then each $|B_j|$ is strictly positive on M_s, see, *e.g.*, [Ga, Chapter X]. Assume that $f_1,\dots,f_m$, $f\in S(A)$, where $A\subset C(X)$ is self-adjoint, satisfy for some $c,\delta>0$

$$\max_{1\le j\le m}|f_j(w)|\ge\delta\quad\text{for all } w\in M\big(S(A)\big);\tag{1.2}$$

$$|f(z,x)|\le c\cdot\omega\Big(\max_{1\le j\le m}|B_j(z)f_j(z,x)|\Big)\quad\text{for all } (z,x)\in\mathbf{D}\times X.\tag{1.3}$$

Then f belongs to the ideal $I\subset S(A)$ generated by $B_1f_1,\dots,B_mf_m$.

In fact, the pullbacks of $B_1,\dots,B_m$ to $M(H^\infty)\times X$ satisfy condition (B) of Theorem 1.11. This and (1.2) imply that $B_1f_1,\dots,B_mf_m$ satisfy this condition as well. Thus the required result follows from the theorem.

Remark 1.13. It is still unclear whether the conclusion of the theorem is valid under assumption (A) only, and whether the condition $\lim\limits_{t\to 0^+}\frac{\omega(t)}{t^3}=0$ can be replaced by $\lim\limits_{t\to 0^+}\frac{\omega(t)}{t^2}=0$; *cf.* [Ga, Chapter VIII.2] for a similar problem. (Note that it cannot be replaced by $\lim\limits_{t\to 0^+}\frac{\omega(t)}{t^\alpha}=0$ for $\alpha<2$. Indeed, if B_1, B_2 are interpolating Blaschke products such that $\inf_{z\in\mathbf{D}}\big(|B_1(z)|+|B_2(z)|\big)=0$, then $|B_1(z)B_2(z)|\le\max\{|B_1^2(z)|,|B_2^2(z)|\}$, $z\in\mathbf{D}$, and $\max\{|B_1^2|,|B_2^2|\}$ is strictly positive on M_s, but B_1B_2 does not belong to the ideal $I\subset H^\infty$ generated by B_1^2, B_2^2, see [R].)

Let $S_N(H^\infty):=S(H^\infty;\dots;H^\infty)$ be the N-dimensional slice algebra on $M(H^\infty)^N$ of continuous functions f such that $f(x,\cdot,y)\in H^\infty$ for each $x\in M(H^\infty)^{k-1}$ and $y\in M(H^\infty)^{N-k}$, $k=1,\dots,N$. A major open problem posed in the mid-1960s asks whether the maximal ideal space of $S_N(H^\infty)$ is $M(H^\infty)^N$, see, *e.g.*, [Cu] and references therein. This fact is obtained now as a corollary of Theorem 1.10.

Corollary 1.14. *$M\big(S_N(H^\infty)\big)=M(H^\infty)^N$.*

1.4. In this subsection we apply Theorem 1.10 to the study of certain operator corona problems, analogs of the Sz.-Nagy problem [SN] on existence of a bounded holomorphic left inverse to an H^∞ function on $\mathbf{D}$ with values in the space of bounded linear operators between two separable Hilbert spaces. (For recent developments in the Sz.-Nagy problem see, *e.g.*, [T1], [T2], [TW], [V] and references therein. In `arXiv:1103.5237`, using the results of the present paper, we solve the generalized Sz.-Nagy problem for operator-valued holomorphic functions on $\mathbf{D}$ with relatively compact images.)

To formulate the result we recall the definition of covering dimension:

For a normal space X, $\dim X\le n$ if every finite open cover of X can be refined by an open cover whose order $\le n+1$. If $\dim X\le n$ and the statement $\dim X\le n-1$ is false, we say $\dim X=n$. For the empty set, $\dim\varnothing=-1$.

For a commutative complex unital Banach algebra A by $M^{k,n}(A)$ we denote the space of $k\times n$ matrices, $k\le n$, with entries in A.

THEOREM 1.15. *Let $F = (f_{ij})\colon \mathbf{D} \to M^{k,n}(A)$, $k < n$, be such that each $f_{ij} \in H^\infty_{\mathrm{comp}}(A)$. Assume that there exists $\delta > 0$ such that the family $h_1, \ldots, h_\ell \in H^\infty(A)$ of minors of F of order k satisfies the corona condition:*

$$\sum_{i=1}^{\ell} \big|\varphi\big(h_i(z)\big)\big| \geq \delta \quad \text{for all } z \in \mathbf{D} \text{ and } \varphi \in M(A). \tag{1.4}$$

If $M(A)$ is the inverse limit of a family of compact Hausdorff spaces $\{M_\alpha\}_{\alpha\in\Lambda}$ such that each M_α is homotopically equivalent to a metrizable compact space X_α with $\dim X_\alpha \leq d$ and if $n-k-1 \geq \lfloor \frac{d}{2} \rfloor$ for $d \geq 3$, and $n-k-1 \geq 0$ otherwise, then there exists a matrix-function $\widetilde{F} = (\widetilde{f}_{ij})\colon \mathbf{D} \to M^{n,n}(A)$ with entries in $H^\infty_{\mathrm{comp}}(A)$ such that $\det \widetilde{F} = 1$ and $\widetilde{f}_{ij} = f_{ij}$ for $1 \leq i \leq k$, $1 \leq j \leq n$.

REMARK 1.16. (a) According to the result of Mardešić [M] any compact Hausdorff space X with $\dim X \leq d < \infty$ can be presented as the inverse limit of a family of metrizable compact spaces X_α with $\dim X_\alpha \leq d$. In particular, Theorem 1.15 holds under the assumptions that $\dim M(A) \leq d$ and $n-k-1 \geq \lfloor \frac{d}{2} \rfloor$ for $d \geq 3$, and $n-k-1 \geq 0$ otherwise.

(b) Theorem 1.15 provides conditions for an extension of F up to an invertible bounded holomorphic $M^{n,n}(A)$-valued function. This, in particular, implies that under the conditions of the theorem, F is left invertible in the considered class.

A Banach algebra $H^\infty_{\mathrm{comp}}(A)$ ($\cong \mathcal{O}\big(M(H^\infty); A\big)$) for which Theorem 1.15 is valid for all $1 \leq k < n < \infty$ is called *Hermite*. (An equivalent definition is that every finitely generated stably free $H^\infty_{\mathrm{comp}}(A)$-module is free.)

A Banach algebra $H^\infty_{\mathrm{comp}}(A)$ is called *projective free* if for any $n \in \mathbf{N}$ every idempotent matrix $F\colon \mathbf{D} \to M^{n,n}(A)$ (*i.e.*, such that $F^2 = F$) with entries in $H^\infty_{\mathrm{comp}}(A)$ is conjugate by means of an invertible matrix $H\colon \mathbf{D} \to M^{n,n}(A)$ with entries in $H^\infty_{\mathrm{comp}}(A)$ to a constant (idempotent) matrix. (An equivalent definition is that every finitely generated projective $H^\infty_{\mathrm{comp}}(A)$-module is free.)

Note that any projective free algebra is Hermite.

THEOREM 1.17. *If the maximal ideal space $M(A)$ of A is the inverse limit of a family of metrizable contractible compact spaces, then the algebra $H^\infty_{\mathrm{comp}}(A)$ is projective free.*

EXAMPLE 1.18. (1) In [S1] Suárez proved that $\dim M(H^\infty) = 2$. Therefore by Corollary 1.14, $\dim S_N(H^\infty) = \dim M(H^\infty)^N = 2N$. It is known that H^∞ is projective free, see, *e.g.*, [Q]. Applying Theorem 1.15 we obtain

(a) $S_2(H^\infty)$ is Hermite;

(b) if $N \geq 3$, then for a matrix $F = (f_{ij})\colon M\big(S_N(H^\infty)\big) \to M^{k,n}(\mathbf{C})$, $n-k \geq N$, with entries in $S_N(H^\infty)$ whose family of minors $h_1, \ldots, h_\ell$ of order k satisfies

$$\sum_{i=1}^{\ell} |h_i(z)| > 0 \quad \text{for all } z \in M\big(S_N(H^\infty)\big),$$

there exists a matrix $\widetilde{F} = (\widetilde{f}_{ij})\colon \mathbf{D} \to M^{n,n}(\mathbf{C})$ with entries in $S_N(H^\infty)$ such that $\det \widetilde{F} = 1$ and $\widetilde{f}_{ij} = f_{ij}$ for $1 \le i \le k$, $1 \le j \le n$.

(2) Let G be a compact connected abelian topological group. A discrete (additive) semigroup $\Sigma_* \subset \widehat{G}$ (:= the Pontriagin dual group of G) is called *pointed* if $0 \in \Sigma_*$ and if $\pm x \in \Sigma_*$ implies that $x = 0$. (It is known that one can introduce an order ≥ 0 on $\widehat{G}$; then, *e.g.*, $\Sigma_+ := \{x \in \widehat{G}; x \ge 0\}$ and $\Sigma_- := \{x \in \widehat{G}; -x \ge 0\}$ are pointed semigroups.)

Let $W_{\Sigma_*}(G) \subset C(G)$ be the Wiener algebra of functions with Bohr–Fourier spectra in Σ_*. It was shown in [BRS] that the maximal ideal space of $W_{\Sigma_*}(G)$ is the inverse limit of a family of metrizable contractible compact spaces. Hence, Theorem 1.17 implies that $H^\infty_{\text{comp}}\big(W_{\Sigma_*}(G)\big)$ is projective free. In particular, $H^\infty_{\text{comp}}\big(W_{\Sigma_*}(G)\big)$ is Hermite.

1.5. Finally, we reformulate the problem on the approximation property for H^∞ in local terms.

For an open subset $U \subset M(H^\infty)$ by $H^\infty(U)$ we denote the Banach space of bounded holomorphic functions on U with supremum norm. According to Proposition 1.3, $H^\infty(U)$ is isomorphic to $H^\infty(U \cap \mathbf{D})$.

THEOREM 1.19. *Let $U \Subset V \subset M_a$ be open subsets and B be a complex Banach space.*

For any function $f \in \mathcal{O}(V; B)$ its restriction $f|_U$ belongs to $H^\infty(U) \otimes B$.

In particular, H^∞ has the approximation property iff for every complex Banach space B and any function $f \in \mathcal{O}\big(M(H^\infty); B\big)$ there exists an open cover $(U_i)_{i\in I}$ of the set M_s of trivial Gleason parts such that $f|_{U_i} \in H^\infty(U_i) \otimes B$ for all $i \in I$.

1.6. Further results. Let R be a Caratheodory hyperbolic Riemann surface and A be a commutative unital complex Banach algebra. By $H^\infty_{\text{comp}}(R; A)$ we denote the Banach algebra of holomorphic functions on R with relatively compact images in A equipped with norm $\|f\| := \sup_{z\in R} \|f(z)\|_A$, $f \in H^\infty_{\text{comp}}(R; A)$. Let $p\colon \mathbf{D} \to R$ be the universal covering of R. The fundamental group $\pi_1(R)$ acts on $\mathbf{D}$ by Möbius transformations. Assume that R satisfies:

(P) There exists a function $h \in H^\infty$ such that $\sup_{z\in\mathbf{D}}\big(\sum_{g\in\pi_1(R)} |h(gz)|\big) < \infty$ and $\sum_{g\in\pi_1(R)} h(gz) = 1$ for all $z \in \mathbf{D}$.

This h determines a linear continuous map $P_A\colon H^\infty_{\text{comp}}(A) \to H^\infty_{\text{comp}}(R; A)$ such that

$$P_A(f_1 \cdot p^* f_2) = P_A(f_1) \cdot f_2 \quad \text{for all } f_1 \in H^\infty_{\text{comp}}(A), f_2 \in H^\infty_{\text{comp}}(R; A),$$

given by the formula

$$P_A(f) := p_*\Big(\sum_{g\in\pi_1(R)} h(gz) f(gz)\Big), \quad f \in H^\infty_{\text{comp}}(A); \tag{1.5}$$

here $p_*\colon p^*\big(H^\infty_{\text{comp}}(R; A)\big) \to H^\infty_{\text{comp}}(R; A)$ is inverse to the pullback map p^*.

Existence of such h for R a finite bordered Riemann surface was established in [Fo], for R a homogeneous Denjoy domain in [C2] (see also [JM] for a more general setting), and for R a subdomain of an unbranched covering R' of a finite bordered Riemann surface such that inclusion $R \hookrightarrow R'$ induces a monomorphism of the fundamental groups in [Br2]. In all these cases taking the pullback by p of functions from $H^\infty_{\rm comp}(R;A)$, then solving the corresponding problem in $H^\infty_{\rm comp}(A)$ and applying to the solution the map P_A we obtain analogs of Theorems 1.7, 1.9 and 1.10 on R (with H^∞ replaced by $H^\infty(R)$ and $H^\infty_{\rm comp}(A)$ replaced by $H^\infty_{\rm comp}(R;A)$ in the original versions of these theorems). In fact, similar results are valid on a Riemann surface R of finite type (since it is biholomorphic to a bordered Riemann surface with finitely many points removed). In this case arguing as in [Br2] one obtains that $\dim M\big(H^\infty(R)\big) = 2$. In particular, analogs of Theorems 1.15 and 1.17 are valid on R.

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