C. R. Math. Rep. Acad. Sci. Canada Vol. **33** (4) 2011, pp. 107–115

Invariant Operators with Complex Potentials

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Presented by George Elliott, FRSC

Abstract. For an operator D a potential Q is null isospectral if $\mathrm{Spec}(D+Q) = \mathrm{Spec}(D)$. In this paper we study bundle maps Q on a homogeneous vector bundle over a homogeneous space that are null isospectral potentials for a large class of invariant operators D, including elliptic self-adjoint differential operators. This result includes and generalizes previous results where D is the Laplace operator and Q is a complex valued function rather than a bundle map. To illustrate these results we give the examples of the Laplace and Dirac operators on S^1.

Résumé. Soit Δ l'opérateur de Laplace sur une variété différentiable homogène compacte. On sait qu'il existe des fonctions Q non-nulles à valeurs complexes telles que Δ et $\Delta + Q$ ont le même spectre. Nous montrons ici que ce résultat s'étend largement. Si D désigne un opérateur elliptique invariant et auto-adjoint opérant sur les fonctions à valeurs dans un fibré vectoriel complexe homogène $\mathbf{E}$ il exist des fonctions Q non-nulles à valeurs dans le fibré $\mathrm{End}(\mathbf{E})$ telles que D et $D + Q$ ont le même spectre. Les fonctions Q possibles se laissent calculer facilement dans le cas de S^1 et l'opérateur de Dirac.

1. Introduction.

Let M be a homogeneous space for a compact semi-simple Lie group G and $\boldsymbol{E}$ a homogeneous vector bundle over M, with fibre a complex vector space. The purpose of this paper is to give conditions on a bundle map $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)$ so that $D + Q$ and D are isospectral for all suitable homogeneous operators D on $\mathbf{E}$, in particular for all self-adjoint elliptic operators.

If $\mathbf{E}$ is such a homogeneous bundle over G/H let $L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$ denote the subspace of $L^2\big(\mathrm{End}(\mathbf{E})\big)$ spanned by the highest weight vectors, excluding those for trivial ones, for the action of G. The suitable operators are those satisfying Condition 2.2, and are also homogeneous. This class includes elliptic self-adjoint differential operators. Notice that if $Q \neq 0$ then $D+Q$ is not homogeneous. The main result is as follows.

Theorem 1.1. *If $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$ then* $\mathrm{Spec}(D+Q) = \mathrm{Spec}(D)$. *For any eigenvalue the eigenspaces may be generalized eigenspaces with those for D and $D+Q$ having the same dimension.*

Received by the editors on February 16, 2011.
The first author was supported by the C. C. Hsiung fund.
AMS Subject Classification: Primary: 58G25; secondary: 57S25.

The operator $D + Q$ is not self-adjoint, but has a beautiful and complete spectral theory (see [3]). One important difference from the self-adjoint case is that we now have generalized eigenfunctions rather than ordinary ones.

Previously, in [4], the case of the Laplacian acting on complex valued functions with a complex valued function as a potential was considered. In [2] the bundle case was considered with a potential obtained from the group algebra. Both of these isospectral results are special cases of the results in this paper. The argument used in this paper closely follows that given by V. Guillemin and A. Uribe in [4], while extending it to a significantly more general case.

The most natural operator to study after the Laplacian is, probably, the Dirac operator. Using detailed results for the spectrum of the Dirac operator, see [1], the result for the Laplacian was initially extended to the Dirac operator. However, it was clear that the result held for a wider class of operators, and was more properly viewed as depending on the bundle rather than the operator. The potentials found here are null isospectral for all operators on the given bundle that satisfy the conditions on the spectrum. Following the terminology of [4] (with some misgivings on the part of the first author) we call $L^2\big(\mathrm{End}(\mathbf{E})\big)_0^h$ Fegan potentials for $\mathbf{E}$. In view of the example given in Section 4, for any particular operator it is possible that the null isospectral potential for that operator forms a larger class than is given here. However, such additional potential may well not be null isospectral for all operators.

In Section 2 we summarize some well-known results on the tensor product of representation spaces, and study the potentials on highest weight spaces. The main result is for $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)_0^h$.

Theorem 1.2. *If $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)_0^h$, then the highest weight space $L^2(\mathbf{E})^h$ is invariant under Q and Q is triangular on this space.*

This is given as Lemma 2.3 and the terminology is explained in more detail. Heuristically, using a basis of invariant sections the operator D can be thought of as represented by a diagonal infinite matrix and Q by an upper triangular matrix.

The passage from the highest weight spaces to the whole bundle is carried out in Section 3. In addition to showing the eigenvalues are the same, we also show that so are their multiplicities.

In Section 4 we consider the special case when $M = G$ and the bundle $\mathbf{E}$ is given by a unitary representation $\pi\colon G \to \mathrm{Aut}(E)$ on a complex vector space. We can identify some elements of $L^2\big(\mathrm{End}(\mathbf{E})\big)_0^h$. Let $L^2(G)^\sigma$ denote a subspace of functions with suitably large highest weights, with the details given in Section 4.

Theorem 1.3. *If $Q = \sum q_i T_i \in L^2(G)^\sigma \otimes \mathrm{End}(E)$, with the sum over $i \in \mathbb{N}$, then $\mathrm{Spec}(D + Q) = \mathrm{Spec}(D)$, and the eigenspaces have the same dimension.*

Following this result we conclude with the example of the Dirac operator on S^1, where the bundle map Q is in fact just a complex valued function q. In [4] the example of the Laplacian on S^1 was computed explicitly, using the elementary theory of "regular singular points". A similar calculation is given, but for the Dirac operator it uses the even more elementary method of separation of variables. However, the results are strikingly different. For the Laplacian, $\mathrm{Spec}(\Delta+q) = \mathrm{Spec}(\Delta)$ if and only if q is either a Fegan potential or its conjugate. For the Dirac operator, $\mathrm{Spec}(D+q) = \mathrm{Spec}(D)$ if the function q has integer constant term in its Laurent expansion. Thus the Dirac operator has a larger class of null isospecral potentials than the Laplacian. While the inverse problem is understood for these operators on S^1, it is still open on other spaces.

2. **Operators on highest weight spaces.** Let V_λ and V_μ be irreducible representations of a semi-simple compact group G with highest weights λ and μ. Let β be the weights of V_λ with weight vectors v_β, and α be the weights of V_μ with weight vectors v_α. Here each weight is repeated as many times as its multiplicity and the weight vectors are chosen to be a basis for the weight space, so for repeated weights they are the various distinct elements of this basis.

The tensor product decomposes into a sum of irreducible representations: $V_\lambda \otimes V_\mu = \sum_\nu m(\lambda\otimes\mu,\nu)V_\nu$, where $m(\lambda\otimes\mu,\nu)$ is the multiplicity of the representation with highest weight ν in the tensor product $V_\lambda\otimes V_\mu$. The weight vectors are $v_\beta\otimes v_\alpha$ with weight $\beta+\alpha$. The problem is to identify which of these are highest weight vectors of irreducible representations. This was done implicitly in [5] and explicitly in [6]. The complete solution involves some delicate combinatorial considerations and the reader is referred to the references for the details. However, there are some statements that are fairly easy to see.

Let $m_\tau(\sigma)$ be the multiplicity of the weight σ in the irreducible representation with highest weight τ. The highest weights of the irreducible representations in the sum are all of the form $\nu = \lambda + \alpha$, and the highest weight vectors are $v_\nu = v_\lambda + \alpha = v_\lambda \otimes v_\alpha$. Note that in the first summand, λ, is the highest weight and the first factor, v_λ, is the highest weight vector of V_λ. This does not say that all weights of this form are highest weights of representations in the sum. However, in the special case when $\lambda+\alpha$ is dominant for λ, the highest weight of V_λ and *all* weights α of V_μ, then all of these do in fact occur, and furthermore $m(\lambda\otimes\mu,\lambda+\alpha) = m_\mu(\alpha)$.

If λ is sufficiently near to the walls of the Weyl chamber, $\lambda+\alpha$ is not dominant for all α and not all of them occur. Obviously, if $\lambda+\alpha$ is not dominant it cannot occur, $m(\lambda\otimes\mu,\lambda+\alpha) = 0$. However, some of the dominant $\lambda+\alpha$ may not occur owing to a "cancellation" with a nondominant weight. Thus $m(\lambda\otimes\mu,\lambda+\alpha)$ need not equal $m_\mu(\alpha)$ and may even be zero. In all cases the weight $\lambda+\mu$ does occur with multiplicity one and this representation has highest weight vector $v_\lambda\otimes v_\mu$.

Let H be a closed subgroup of the compact semi-simple group G, then $M = G/H$ is a homogeneous space. If $\pi\colon H \to \mathrm{Aut}(E)$ is a representation on a complex vector space E then construct the homogeneous bundle $\mathbf{E} = G\times_H E$ over M with fibre E. Denote the L^2 sections of $\mathbf{E}$ by $L^2(\mathbf{E})$. The left action of G on itself induces an action on $L^2(\mathbf{E})$ denoted by ν_s. Decompose the space of sections into isotypic components

$$L^2(\mathbf{E}) = \sum E_\lambda, \tag{2.1}$$

where each E_λ consists of multiple copies of the irreducible representation V_λ with highest weight λ and is finite dimensional. The sum is an L^2 Hilbert space completion.

DEFINITION 2.1. For any representation U of G, let U^h be the subspace of U spanned by the highest weight vectors.

CONDITION 2.2. Require that the spectrum of D is discrete, with $\pm\infty$ as the only accumulation points, and has finite dimensional generalized eigenspaces that span a dense subset the space of sections $L^2(\mathbf{E})$.

Note that an elliptic self-adjoint differential operator D satisfies this condition. However, other operators, for example pseudodifferential, may also satisfy the condition.

Let D be a homogeneous operator on $\mathbf{E}$. Then D is equivariant under the action of G and so each E_λ is invariant under D. Further the highest weight spaces E^h_λ and $L^2(\mathbf{E})^h$ are both invariant under D.

A *potential* on $\mathbf{E}$ is $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)$. Thus there is an operator $D + Q$ on $L^2(\mathbf{E})$. Restrict the choice of Q to $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$, that is Q is a highest weight potential with no component in the trivial ($\lambda = 0$) representation space.

LEMMA 2.3. *If $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$, then*
(a) $Q\big(L^2(\mathbf{E})^h\big) \subset L^2(\mathbf{E})^h$, *and*
(b) $Q(\mathbf{E}^h_\lambda) \subset \sum_{\mu>\lambda} E^h_\mu$.

PROOF. Decompose $L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0 = \sum U^h_\xi$ into isotypic components. Then $Q = \sum Q_\xi$, where $Q_\xi \in U^h_\xi$. If $s \in E^h_\lambda$, then

$$Q(s) \in \sum U^h_\xi \otimes E^h_\lambda. \tag{2.2}$$

By the results on the tensor product of representation spaces summarized earlier in this section, we have $V^h_\xi \otimes V^h_\lambda \subset V^h_{\xi+\lambda}$. Thus $Q(s) \in \sum V^h_{\xi+\lambda}$, where the terms are repeated as many times as the multiplicities require. This proves (a). Further, setting $\mu = \xi + \lambda$ gives $Q(E^h_\lambda) \subset \sum E^h_\mu$ for $\mu > \lambda$, proving (b). □

REMARK 2.4. Part (a) states that Q preserves the highest weight spaces. Part (b) states that Q increases the weight of the representation space when applied to a section. Notice that the range of the summation in (b) is over μ strictly greater than λ. This property is referred to as "Q is triangular".

3. Spectra with potentials. The space $L^2(\mathbf{E})^h$ is invariant under both D and $D+Q$. Since D is self-adjoint it has a complete and well-known spectral theory: the eigenvalues of D are real and discrete, and the eigenvectors span a dense subset of $L^2(\mathbf{E})^h$ and the eigenspaces are finite dimensional. Further, the eigenvectors are ordinary eigenvectors, that is $Ds = \alpha s$. The operator $D+Q$ is not self-adjoint but still has a complete spectral theory. This is given in [3]. In the present case the result, found in chapter V, section 10, is as follows.

PROPOSITION 3.1. *If D satisfies Condition 2.2, then*
(a) *The spectrum of $D+Q$ is discrete and only has $\pm\infty$ as points of accumulation.*
(b) *For every $\varepsilon > 0$, all but finitely many eigenvalues, β, lie in the sectors $-\varepsilon < \arg\beta < \varepsilon$ and $\pi - \varepsilon < \arg\beta < \pi + \varepsilon$.*
(c) *For an eigenvalue β, the generalized eigenspace is finite dimensional. The generalized eigenspace is the set of all sections s such that $(D+Q-\beta)^k s = 0$ for some positive integer k.*
(d) *The generalized eigenspaces span a dense subset of $L^2(\mathbf{E})^h$.*

REMARK 3.2. This is stated for operators on $L^2(\mathbf{E})^h$; it also holds for operators on $L^2(\mathbf{E})$.

THEOREM 3.3. $\mathrm{Spec}(D+Q) \subset \mathrm{Spec}(D)$ *on* $L^2(\mathbf{E})^h$.

PROOF. Let s be an eigensection for $D+Q$ with eigenvalue β. Now using the decomposition of $L^2(\mathbf{E})^h = \sum E_\lambda^h$ we can write s as a sum of components

$$s = \sum s_\lambda. \tag{3.1}$$

Call the lowest weights in equation (3.1) the "lowest highest weights", using the partial ordering $\mu > \lambda$ if and only if $\mu - \lambda$ is positive. Applying $D+Q$ to s_λ, and noting that s_λ is an eigensection of D, gives

$$(D+Q)s_\lambda = Ds_\lambda + \sum_{\mu>\lambda} a_\mu s_\mu = \alpha_\lambda s_\lambda + \sum_{\mu>\lambda} a_\mu s_\mu, \tag{3.2}$$

where α_λ is the relevant eigenvalue of D. Now $(D+Q)s = \beta s$, so by the uniqueness of the expansion $\beta = \alpha_\lambda$ for all α_λ occurring. Thus only one α_λ occurs in the sum of lowest highest weights and the spectrum of $D+Q$ is contained in that of D. □

It is clear from this proof that if s_β is an eigensection for $D+Q$, with corresponding eigenvalue β, then there is an expansion $s_\beta = \sum s_\lambda$, then all the terms with a lowest highest weight are eigensections of D for the same eigenvalue, β.

THEOREM 3.4. $\mathrm{Spec}(D) \subset \mathrm{Spec}(D+Q)$.

PROOF. Let α be an eigenvalue of D. Then by construction there is a corresponding eigensection s_λ. Since the eigensections of $D+Q$ span $L^2(\mathbf{E})^h$, this can be written as an expansion $s_\lambda = \sum s_\beta$.

Consider the lowest highest weight terms of each s_β, then by the uniqueness of the series (and the fact that the action of Q increases weights) one of these is s_λ. As in the proof of the previous theorem, it follows that $\beta = \alpha$ for one of the β and so α is an eigenvalue of $D + Q$. □

Since the group G acts on $L^2(\mathbf{E})$ so does the Lie algebra $\mathfrak{g}$: if $X \in \mathfrak{g}$ then $\nu(X)$ is the action on $L^2(\mathbf{E})$. The complexification of $\mathfrak{g}$ is $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ and, since E is complex, this action extends to one by the complexification:

$$\nu(X + iY)s = \nu(X)s + i\nu(Y)s. \tag{3.3}$$

A homogeneous operator D commutes with the action of G, and hence with the actions of both $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$. Let $\mathfrak{n} \subset \mathfrak{g}_\mathbb{C}$ be the nilpotent part of the complexification, then $u \in L^2(G)^h$ if and only if $\nu(X)u = 0$ for all $X \in \mathfrak{n}$. Since $L^2\big(\mathrm{End}(\mathbf{E})\big) = L^2(G) \otimes_H \mathrm{End}(E)$, this remark generalizes.

PROPOSITION 3.5. *An element $u \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h$ if and only if $\nu(X)u = 0$ for all $X \in \mathfrak{n}$.*

LEMMA 3.6. *For $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$, $D+Q$ commutes with the action of $\mathfrak{n}$ by ν.*

PROOF. As observed above D commutes with this action, so it only remains to show that Q also commutes. Let $s = \sum s_\lambda$ and $Q = \sum Q_\xi$. If $X \in \mathfrak{n}$, then $\nu(X)Q_\xi = 0$ since $Q_\xi \in V^h_\xi$. Then for $X \in \mathfrak{n}$, we calculate:

$$\begin{aligned} \nu(X)(Qs) &= \nu(X)\Big(\sum Q_\xi s\Big) \\ &= \sum \big(\nu(X)Q_\xi\big)s + \sum Q_\xi \nu(X)s \\ &= \sum Q_\xi \nu(X)s \\ &= Q\big(\nu(X)s\big). \end{aligned} \tag{3.4}$$

A key step in passing from $L^2(\mathbf{E})^h$ to $L^2(\mathbf{E})$ is the next lemma. □

LEMMA 3.7. *If $s \in L^2(\mathbf{E})$ is a common eigensection of $\nu(X)$ for all $X \in \mathfrak{n}$, then $s \in L^2(\mathbf{E})^h$.*

PROOF. Since $X \in \mathfrak{n}$ is nilpotent, any eigenvalue of $\nu(X)$ is 0. The result now follows from Proposition 3.5. □

Now to complete the passage from $L^2(\mathbf{E})^h$ to $L^2(\mathbf{E})$.

THEOREM 3.8. $\mathrm{Spec}(D + Q) = \mathrm{Spec}(D)$ *on* $L^2(\mathbf{E})$.

PROOF. Let β be an eigenvalue for $D + Q$ on $L^2(\mathbf{E})$ with corresponding eigenspace E_β. Since $D + Q$ commutes with $\nu_s(X)$ for all $X \in \mathfrak{n}$, E_β is an invariant space of $\mathfrak{n}$. By Lie's theorem, E_β has a common eigenvector s_β and, by the last lemma, $s_\beta \in L^2(\mathbf{E})^h$. Thus β is an eigenvalue of $D + Q$ on $L^2(\mathbf{E})^h$ and hence an eigenvalue of D, since $\mathrm{Spec}(D + Q) = \mathrm{Spec}(D)$ on $L^2(\mathbf{E})^h$, by an earlier result. The reverse inclusion of spectra follows similarly. □

Following the argument of [4], let B be an operator on a Hilbert space H, having a discrete spectrum with $\pm\infty$ as the only accumulation points and such that the generalized eigenvectors span H. Let β be an eigenvalue with corresponding generalized eigenspace H_β, which is required to be finite dimensional. Since the generalized eigenspaces of B span H, pick an orthonormal basis $\{x\}$ of eigenvectors. Define an operator $P_\Gamma \colon H \to H$ by $P_\Gamma u = \frac{1}{2\pi i}\int_\Gamma (z-B)^{-1}u\,dz$, where Γ is a simple closed curve in the complex plane traced in an anticlockwise direction such that no eigenvalue of B lies on it. The trace of P_Γ is $\mathrm{tr}(P_\Gamma) = \sum \langle P_\Gamma x, x\rangle$ with the sum over all x in the basis. For $x \in H_\beta$ of unit length, $\langle (z-B)^{-1}x, x\rangle = (z-\beta)^{-1}$. Thus $\mathrm{tr}(P_\Gamma) = \frac{1}{2\pi i}\sum_\beta \int_\Gamma \frac{1}{(z-\beta)}\,dz$, so by Cauchy's integral formula $\mathrm{tr}(P_\Gamma)$ is the sum of the dimensions of the eigenspaces, $\sum \dim(H_\beta)$, for all β inside the curve Γ. This also shows that P_Γ is of trace class since this sum is over a finite set of eigenvalues.

Apply this to each member of the family of operators $B_t = D + tQ$, where D and Q are as in the previous sections. Let β be an eigenvalue of D and let $2r$ be the distance from β to its closest neighbouring eigenvalue. Since $\mathrm{Spec}(D+tQ) = \mathrm{Spec}(D)$, β is an eigenvalue of B_t for all t. Let Γ be the circle centred at β with radius r. Then, for all t, none of the eigenvalues of B_t lie on Γ and the only eigenvalue inside Γ is β. Define P_t by $P_t x = \frac{1}{2\pi i}\int_\Gamma (z-B_t)^{-1}x\,dz$; then $\mathrm{tr}(P_t)$ is the multiplicity of the eigenvalue β for the operator B_t. Now $\mathrm{tr}(P_t)$ is continuous in t and integer valued, so it is constant and the multiplicities of β as an eigenvalue of both D and $D+Q$ are the same.

Theorem 3.9. *The dimensions of the eigenspaces are equal:* $\dim\big(E_\beta(D+Q)\big) = \dim\big(E_\beta(D)\big)$.

4. The group case.

In this section we consider the special case of $M = G$ and E is a representation space of G. The bundle $\mathbf{E}$ is trivial and so $\mathbf{E} = G \times E$ and $L^2\big(\mathrm{End}(\mathbf{E})\big) = L^2(G) \otimes \mathrm{End}(E)$. Thus a potential decomposes as

$$Q = \sum q_i T_i, \tag{4.1}$$

where the q_i are functions on G, $T_i \in \mathrm{End}(E)$, and the sum over $i \in \mathbb{N}$ converges in the L^2 sense. We introduce the following definition.

Definition 4.1.

(a) A weight μ *dominates* a weight λ if $\mu - \lambda$ is a dominant weight, and *strictly dominates* if $\mu - \lambda$ is a non-zero dominant weight.

(b) A weight μ *dominates* (respectively *strictly dominates*) a representation E if μ dominates (strictly dominates) every weight of E.

Remark. This is a partial ordering different from the usual partial ordering. Usually it is required that $\mu - \lambda$ be positive for $\mu > \lambda$, a less restrictive ordering than that $\mu - \lambda$ be dominant. If μ dominates λ then $\mu > \lambda$ in the usual sense, but the converse need not hold. In the proof of Lemma 2.3 (b), we actually proved that $Q(E^h_\lambda) \subset \sum E^h_\mu$ with the sum over μ dominating λ, but we make no use of this stronger condition.

DEFINITION 4.2. For a representation space E, let $L^2(G)^\sigma$ be the span of the highest weight vectors for those weights that strictly dominate E.

THEOREM 4.3. *If $q_i \in L^2(G)^\sigma$ and $Q = \sum q_i T_i$, then $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$.*

PROOF. If μ strictly dominates λ then

$$V^h_\mu \otimes V_\lambda \subset \sum V^h_{\mu+\alpha}, \tag{4.2}$$

where the sum is over all weights α of V_λ. This follows from the well-known results on tensor products of representations summarized in Section 2. The theorem is now immediate. □

COROLLARY 4.4. *If $q_i \in L^2(G)^\sigma$, then $\mathrm{Spec}(D + \sum q_i T_i) = \mathrm{Spec}(D)$, and the eigenspaces have the same dimension.*

Let $C[G] = \{\sum \alpha_X X : X \in G$ only finitely many $\alpha_X \neq 0\}$ be the group algebra of G. Then if $\pi\colon G \to \mathrm{Aut}(E)$, the linear maps T_i are given by $T_i = \pi(x_i)$ for some $x_i \in C[G]$. Consider potentials of the form

$$Q(x) = \sum_{\lambda \in \Lambda, k \in K} \langle \pi_\lambda(x) v_\lambda, u_{\lambda k}\rangle \pi(x^k), \tag{4.3}$$

here v_λ is the highest weight vector of V_λ (the irreducible representation with highest weight λ), $u_\lambda k$ is a vector in V_λ, Λ is the set of dominant weights, and K is a finite set of integers. Then Theorem 4.3 yields.

THEOREM 4.5. *If λ dominates $k\tau$ for all λ and k occurring in equation* (4.3) *and all highest weights τ of E, then $Q \in L^2\big(\mathrm{End}(\mathbf{E})\big)^h_0$.*

PROOF. The condition λ dominates $k\tau$ implies that highest weights of Q dominate E, and v_λ being a highest weight gives that Q is a sum of highest weight vectors. Notice that E need not be irreducible and so the highest weight vectors, τ, of E need not be unique. □

This is a generalization of [2, Theorem 1.1] since the sum needs only be L^2 convergent rather than finite.

To illustrate, let $G = S^1$ and use the coordinate $z = e^{i\theta}$ regarding S^1 as the unit circle in the complex plane. The spin representation is the trivial action of S^1 on $\mathbb{C}$, the spin bundle is the trivial one dimensional bundle $\mathbf{S} = S^1 \times \mathbb{C}$, and the spinors are just complex valued functions. The Dirac operator is $D = \frac{1}{i}\frac{d}{d\theta} = z\frac{d}{dz}$, which has eigenvalues $n \in \mathbb{Z}$, with multiplicity 1 and corresponding eigenfunctions $z^n = e^{in\theta}$. The potential is then a function

$Q = q = \sum_{n\geq 1} a_n e^{in\theta} = \sum_{n\geq 1} a_n z^n$. To find the eigenvalues of $D + q$, it is necessary to solve the equation $Df + qf = \lambda f$. This is the first order ordinary differential equation $z\frac{df}{dz} = (\lambda - q)f$, which is solved by separating the variables. Let $p = \frac{q}{z} = \sum_{n\geq 1} a_n z^{n-1}$. This is an analytic function with an integral $P = \int p\,dz = \sum_{n\geq 1} \frac{a_n z^n}{n}$. Set $R = ce^{-P}$, for c a constant, then the solution of the equation is $f = Rz^\lambda = cz^\lambda e^{-\int \frac{q}{z}\,dz}$. Since f and q are analytic functions, it follows that $\lambda = n \in \mathbb{Z}$. Thus the eigenvalues of $D + q$ are the integers, each with multiplicity 1, that is, $\mathrm{Spec}(D) = \mathrm{Spec}(D + q)$.

The condition that the Fourier series of q only contains positive powers of $e^{i\theta}$ ensures the regularity of R. It is clear that this is too strong a condition. The same result holds providing the constant term in the Laurent series of q is an integer, and fails if q has a noninteger constant term. This is rather different from the case of the Laplacian on S^1, see [4], when the potential q does not change the spectrum if the Laurent series of q consists of only positive powers or only negative powers, but fails if there are both positive and negative powers.

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