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Homomorphisms from the Fredholm Semigroup to Abelian Semigroups

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Abstract. It is shown that the universal enveloping abelian semigroup of the semigroup of Fredholm operators on an infinite-dimensional Hilbert space is the group of integers.

Résumé. On démontre que l'index de Fredholm est le morphisme universel du semigroupe d'opérateurs de Fredholm sur un espace de Hilbert de dimension infinie dans un semigroupe abélien.

1. It is well known from Atkinson's theorem (see, *e.g.*, [6]) that the Fredholm index for Hilbert space operators factors as a semigroup map through the group of invertible elements of the Calkin algebra. This raises the question of the universality of these various maps: from the Fredholm operators to the group of invertible elements of the Calkin algebra, from this group to the group of integers, and from the Fredholm semigroup to the group of integers.

We shall show in this note that all three of these maps are universal in the appropriate sense, depending on which map one considers.

Theorem.

(i) *The group of invertible elements of the Calkin algebra, with the norm topology, is the universal enveloping Hausdorff topological group of the semigroup of Fredholm operators (on a given Hilbert space), with the norm topology.*

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(ii) *The group of integers is the universal enveloping abelian group, and in fact the universal enveloping abelian semigroup, of the semigroup of Fredholm operators on an infinite-dimensional Hilbert space. (Hence this also holds for the group of invertible elements of the Calkin algebra.)*

Proof. (i) Let us first calculate the purely algebraic enveloping group of the semigroup of Fredholm operators. In fact, as a moment's thought shows, Atkinson's theorem is valid for the semigroup of Fredholm operators on an arbitrary vector space, defined as those linear operators with both kernel and cokernel of finite dimension—and when the vector space is a Banach space the Fredholm operators in the Banach space sense are just the bounded ones, *i.e.*, the Fredholm operators in the algebraic sense which are bounded (since the image of a bounded Banach space operator must be a closed subspace if it has finite codimension). Let us show that the group of invertible elements of the algebra of all linear operators on a vector space, modulo the two-sided ideal of operators of finite rank, is the enveloping group of the semigroup of all Fredholm operators (in the algebraic sense described above, which coincides with the usual sense in the case of bounded operators on a Banach space). Our proof will show at the same time that every semigroup homomorphism of the semigroup of Fredholm operators on a Hilbert space into a group factors uniquely through this algebraic analogue of the Calkin algebra; the purely algebraic enveloping group of the Hilbert space Fredholm semigroup is then seen to be its image in the algebraic Calkin algebra.

We must show that a homomorphism of the semigroup of Fredholm operators on a vector space into a group is invariant under finite-rank perturbations. If the vector space is finite-dimensional, this is trivial. (In this case, 0 is Fredholm, but $0s = 0$ for every s, and so any homomorphism into a group must be constant.) If the vector space is infinite-dimensional, let us consider first the perturbation consisting of subtracting from the identity operator, 1, an idempotent with finite-dimensional range, e. Thus, both 1 and $1 - e$ are Fredholm operators; in fact, there exist Fredholm operators s and t such that $ts = 1$ and $st = 1 - e$; furthermore, if the vector space is a Hilbert space and e is bounded and self-adjoint, then s and t may be chosen to be bounded (of norm one). (If the range of e has dimension one, and, say, the vector space has countable dimension or countable Hilbert space dimension, then s may be taken to be the unilateral shift in a basis in which e projects onto the first basis vector, and t the shift in the opposite direction, killing the first basis vector.) Under a homomorphism into a group, as $ts = 1$ the images of s and t must be the inverses of each other, and so the image of $st = 1 - e$ must also be the identity element of the group. Now, if x is an arbitrary Fredholm operator and a is an operator of finite rank (so that $x + a$ is also Fredholm), then with e an idempotent with range equal to the (finite-dimensional) range of a, we have

$$(1 - e)(x + a) = (1 - e)x + (1 - e)a = (1 - e)x.$$

So, under a homomorphism into a group, $(1 - e)(x + a)$ and $(1 - e)x$ map into the same element, but so also do $1 - e$ and 1, and so $x + a$ and x must also map into the same element, as required.

Let us now compute the enveloping Hausdorff topological group. A continuous homomorphism of the Fredholm semigroup into a topological group must first, as shown above, factor through the algebraic Calkin algebra, *i.e.*, be invariant under finite-rank perturbations. Hence by continuity, since every compact operator on a Hilbert space is a limit of operators of finite rank, if the topology is Hausdorff then the homomorphism is invariant under compact perturbations, and in other words factors through the usual Calkin algebra. (Interestingly, continuity of the inverse operation in the group or even of multiplication is not used, and T_1 separation of the topology is sufficient.)

(ii) Let there be given a homomorphism from the semigroup of Fredholm operators on an infinite-dimensional Hilbert space (or, alternatively, an infinite-dimensional vector space) into an abelian semigroup. Passing to the image of the homomorphism, we may suppose that the homomorphism is surjective and hence in particular that the image of the identity operator is the zero element of the abelian semigroup. (*A priori*, there needn't be a zero element.) Using abelianness of the semigroup instead of the group property, in the proof of the statement (i) above, we see that the images of the Fredholm operators s and t as there, with $ts = 1$ and $st = 1 - e$, where e is an arbitrary finite-dimensional projection, must be the (additive) inverses each of the other (as their sum is the zero element), and in particular the image of $1-e$ must be the same as the image of 1 (namely the zero element). Hence, again as in the proof of (i) above, the given homomorphism must be invariant under finite rank perturbations.

Recall that every Fredholm operator of index 0 is a finite-rank perturbation of an invertible operator. Recall also that with s and t as above, with $ts = 1$ and $st = 1-e$ where now we choose e to be a projection with one-dimensional range, the Fredholm semigroup is generated by s and t together with the Fredholm operators of index 0. (Note that t has index—the dimension of the kernel minus the dimension of the cokernel—equal to 1, and s has index -1. If r has index $n > 0$, then $s^n r$ has index 0, and $r = 1r = (t^n s^n)r = t^n(s^n r)$. If r has index $n < 0$, then $rt^{|n|}$ has index 0, and $r = r1 = r(t^{|n|}s^{|n|}) = (rt^{|n|})s^{|n|}$.)

By [1, Corollary 4.7], every element of the group of invertible bounded operators on an infinite-dimensional Hilbert space is a finite product of commutators. (Inspection of the proof shows that this also is the case for the group of invertible operators on an infinite-dimensional vector space.) The invertible operators are therefore taken into the zero element by any homomorphism of the semigroup of Fredholm operators into an abelian semigroup with zero taking the identity operator into zero. Since, as shown above, such a homomorphism is invariant under finite-rank perturbations, it follows that it takes every Fredholm operator of index 0 into the zero element. Since (as shown) these operators together with s and t generate the whole semigroup of Fredholm operators and since the given homomorphism is now assumed to be surjective, it follows that the images of s and t generate the given abelian semigroup with zero and that the homomorphism is determined by what it does to s and t. As already noted, since $ts = 1$ the images of s and t must be the inverses of each other. It follows that the abelian semigroup is a group, generated by the single element t, and that, furthermore, the given homomorphism factors through the index map, any operator of index n being mapped into n times the image of t. Since the range of the index map is the integers, this shows that the given map factors uniquely through the integers (via the index map), and so the group of integers is the universal enveloping abelian semigroup. (The same proof, as remarked, is applicable to the semigroup of Fredholm operators on an infinite-dimensional vector space. The assumption of infinite-dimensionality is necessary since, in the finite-dimensional case, the determinant is also a semigroup homomorphism.)

(To show that the same conclusion holds for the group of invertible elements of the Calkin algebra, let there be given a homomorphism from this group to an abelian semigroup, and compose this with the canonical map from the Fredholm semigroup to the group of invertible elements of the Calkin algebra. Let us refer to this group as the Atkinson group. This combined semigroup homomorphism must factor uniquely through the canonical map from the Fredholm semigroup to its universal enveloping abelian semigroup, which by the first part of the statement of (ii) is the group of integers. In the proof of this statement it was also shown that the index map has the universal property, and of course such a universal map is unique up to an isomorphism of the integers, and so it follows that the combined map under consideration factors (uniquely) through the index map. (Actually, at this point, one does not need to use the proof given above to conclude that the index map must have the universal property, since when it is factored through the canonical map into the integers with the universal property the resulting map from the integers to the integers must be surjective, like the index map, and therefore an isomorphism.) Since the map from the Fredholm semigroup to the Atkinson group is surjective, it follows that the given map from the Atkinson group to the given abelian semigroup factors uniquely through the index map, considered as a map from the Atkinson group to the integers, as desired. An alternative proof of the statement in question is obtained by applying [3, Théorème 7.5] to the Calkin algebra, to conclude that the connected component of the identity in the Atkinson group is equal to the commutator subgroup, whence any (as we may suppose) surjective homomorphism from the Atkinson group into an abelian semigroup is trivial on the connected component, and so constant on components, and therefore factors uniquely through the index map, the level sets of which on the Atkinson group are known to be the connected components.) □

2. The C*-algebra of bounded operators on a Hilbert space is the multiplier algebra of the C*-algebra K of compact operators, and it is possible to generalize Theorem 1 in this direction as follows.

Theorem. *Let A be a stable C*-algebra; this means that A is isomorphic to the C*-algebra tensor product $A \otimes K$ of A with K. Assume that A has an approximate unit consisting of projections. (It would of course be desirable to remove this technical assumption;* cf. *Remark* 3 *below.) Consider the multiplier C*-algebra $M(A)$ of A, and let us define the Fredholm semigroup relative to*

A as the semigroup of elements of $M(A)$ with invertible image in the quotient $M(A)/A$, and the Fredholm index map relative to A as the map from this semigroup to $\mathrm{K}_0(A)$ obtained by mapping to $M(A)/A$ and applying the index map $\mathrm{K}_1(M(A)/A) \to \mathrm{K}_0(A)$.

(i) *The group of invertible elements of the corona algebra $M(A)/A$, with the norm topology, is the universal enveloping Hausdorff topological group of the Fredholm semigroup relative to A, with the norm topology.*

(ii) *The group $\mathrm{K}_0(A)$ is the universal enveloping abelian semigroup of the Fredholm semigroup relative to A. (Hence this also holds for the group of invertible elements of $M(A)/A$.)*

Proof. (i) As in the proof of Theorem 1(i), it suffices to show that if e is a projection in A, then a semigroup homomorphism from the Fredholm semigroup relative to A into a group must agree on the elements $1-e$ and 1. To this end, as before, given a projection e in A it is sufficient to find elements s and t of $\mathrm{M}(A)$ such that $ts = 1$ and $st = 1-e$. Consider $\mathrm{M}(K)$ embedded into $\mathrm{M}(A)$ (as $1\otimes\,\mathrm{M}(K)$) via an isomorphism of A with $A \otimes K$. Modifying this isomorphism by an inner automorphism of (the unitization of) $A \otimes K$, we may suppose that e belongs to $A \otimes M_n \otimes e_{11} \subseteq A \otimes K$ after some identification of K with $\mathrm{M}_n \otimes K$ (e_{11} being the upper left-hand corner matrix unit in K). Again since $\mathrm{M}_n \otimes K$ is isomorphic with K, we may suppose that $e \in A \otimes e_{11} \subseteq A \otimes K$, say $e = p \otimes e_{11}$. Then, considering the sequence of projections $p \otimes e_{11}$, $p \otimes e_{22}$, $p \otimes e_{33}$, *etc.*, and the canonical partial isometries from the first to the second, the second to the third, and so on, adding these up in $\mathrm{M}(A \otimes K)$ (in the strict topology), and adding to the resulting partial isometry the projection $1 - \sum p \otimes e_{ii}$, we obtain an isometry in $\mathrm{M}(A)$ with cokernel projection e; in other words, this isometry and its adjoint satisfy the requirements for s and t.

In fact, the construction above and the associated technical hypothesis concerning the existence of projections are not needed until the proof of the statement (ii). The fact that $x + a$ and x must map into the same element if this is true after they are multiplied on the left by $1 - e$ (see proof of Theorem 1(i)) holds just by cancellation in the codomain. With this approach, e just needs to be an element of A such that ea is equal to a, and one uses that the set of elements x such that there exists such an e is dense in A.

(ii) As in the proof of Theorem 1(ii) it suffices to show, beyond what was shown just above, first, that every element of the Fredholm semigroup relative to A of index 0 can be perturbed by an element of eA for some projection e in A in order to be invertible in $\mathrm{M}(A)$, and, secondly, that every element of the group of invertible elements of $\mathrm{M}(A)$ is a finite product of commutators (*i.e.*, that this group is perfect). The second statement is a special case of [3, Théorème 7.5] (given that, since A is isomorphic to $A \otimes K$, the multiplier algebra $\mathrm{M}(A)$ contains a unital copy of the type I factor $\mathrm{M}(K)$). To check the first statement, let x be a Fredholm element of $\mathrm{M}(A)$ of index 0, equivalently, such that $x + a$ is invertible for some $a \in A$. This equivalence is proven as follows: by the proof of [4, Lemma 2.6], using only that A has an approximate unit consisting of projections, every unitary in $\mathrm{M}(A)/A$ lifts to a partial isometry in $\mathrm{M}(A)$. In other words, passing, as we may, to the case that the image of x in $\mathrm{M}(A)/A$ is unitary, we may suppose that $x_1 = x + a_1$ is a partial isometry in $\mathrm{M}(A)$ for some $a_1 \in A$; the kernel and cokernel projections e and f of x_1 belong to A, and have the same class by hypothesis in $\mathrm{K}_0(A)$; since A is stable, enlarging e and f if necessary, we may suppose they are equivalent, and with a_2 a partial isometry between them, $x + a$ with $a = a_1 + a_2$ is unitary, as desired. (We have proved that $\mathrm{M}(A)/A$ is K_1-injective.) Now, choose a projection e in A such that ea is sufficiently close to a that the element $x+ea$ of $\mathrm{M}(A)$ is invertible (recall that $x + ea$ of $\mathrm{M}(A)$ the set of invertible elements of the Banach algebra $\mathrm{M}(A)$ is open).

Let us review how the assertion (ii) can be deduced from the facts now assembled. Let there be given a semigroup homomorphism from the Fredholm semigroup relative to A to an abelian semigroup.

First, for any projection e in A, with s and t in $\mathrm{M}(A)$ as in the proof of (i), above, *i.e.*, such that $ts = 1$ and $st = 1 - e$, since we may suppose, as earlier, passing to the image of the map, that the map is surjective and the image of 1 is a zero element for the abelian semigroup, we have that the images of s and t are the inverses of each other, and in particular the images of ts and st, *i.e.*, of 1 and $1 - e$, are equal.

Second, for any Fredholm element x of $\mathrm{M}(A)$ of index 0, *i.e.*, by the first paragraph above, such that, for some element ea of eA with $e \in A$ a projection, the element $x + ea$ is invertible, on the one hand (again by the first paragraph) $x + ea$ is a finite product of commutators in the group of invertible elements of $\mathrm{M}(A)$ and so maps into the zero element of the abelian semigroup, and on the other hand the three elements $x + ea$, $(1 - e)(x + ea) = (1 - e)x$, and x all map into the same element, since 1 and $1 - e$ do. In other words, the given homomorphism is zero on the subset of Fredholm elements of index 0.

The conclusion, as we shall now show, follows: a given homomorphism from the Fredholm semigroup relative to A into an abelian semigroup must factor, uniquely, through the index map.

Uniqueness of such a factorization follows from surjectivity of the index map $\mathrm{K}_1\big(\mathrm{M}(A)/A\big) \to \mathrm{K}_0(A)$. (This holds by the six-term exact sequence of Bott periodicity as $\mathrm{K}_0\big(\mathrm{M}(A)\big) = 0$, which follows, for instance, from the fact established in [5, proof of Theorem 6] that, if p and q are orthogonal projections in $\mathrm{M}(A)$ (with A stable) with p equivalent to 1, then also $p + q$ is equivalent to 1.) Actually, since we are only looking at invertible elements of $\mathrm{M}(A)/A$, not of matrix algebras over this algebra, we must check that these invertible elements exhaust the K_1-group (K_1-surjectivity). This is true whenever an infinite type I factor, or, more generally, the Cuntz algebra O_2, is unitally embeddable in a given C*-algebra. (Cf. Theorem 1.9 of [2]. More specifically, if B is such a C*-algebra, then there is an isometry in $\mathrm{M}_2(B)$ with range projection $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, and like any partial isometry in a C*-algebra, this extends to a unitary element of the two-by-two matrix algebra over the C*-algebra in question, *i.e.*, $\mathrm{M}_4(B)$ in this case. This unitary then transforms the canonical images in $\mathrm{M}_4(B)$ of invertible elements in $\mathrm{M}_2(B)$ into the canonical images in $\mathrm{M}_4(B)$ of invertible elements of $B = \mathrm{M}_1(B)$, of course in a way preserving the K_1-class.) If A is non-zero, then, as A is stable, $\mathrm{M}(A)$ unitally contains an infinite type I factor and hence the quotient $\mathrm{M}(A)/A$ unitally contains either this or the Calkin algebra and in either case O_2.

Existence of such a factorization is seen as follows: we must show that if two Fredholm elements of $\mathrm{M}(A)$ have the same index in $\mathrm{K}_0(A)$, then the given semigroup homomorphism agrees on them. Let x and y be such elements, and note that x^*y has index 0. (This follows from polar decomposition for invertible elements of $\mathrm{M}(A)/A$.) Therefore, as in the proof of Theorem 1(ii) above, the given homomorphism takes x^*y into the zero element of the given abelian semigroup (now with a zero element as we have passed to the image of the homomorphism). By the homomorphism property, it follows on the one hand that the image of $xx^*y = x(x^*y)$ is equal to the image of x, and on the other hand (since $xx^*y = (xx^*)y$ and also xx^* has index 0) that the image of xx^*y is equal to the image of y.

(The final statement, concerning the enveloping abelian semigroup of what might be called the generalized Atkinson group relative to the stable C*-algebra A, is proved in the same way as the analogous statement of Theorem 1, with the exception that one now has to use what was shown in the proof of the first part of the statement (ii), namely, that the index map on the Fredholm semigroup relative to A in fact has the universal property, with respect to maps into abelian semigroups. (It does not appear to be possible just to deduce this from the first part of the statement (ii).) An alternative proof, using [3, Théorème 7.5], and avoiding the technical assumption on the C*-algebra, obtains as before.) □

3. Remark The assumption concerning the existence of projections in Theorem 2 may be removed at the cost of weakening the statement in two ways. First, the Fredholm semigroup with respect to a given stable C*-algebra A should be enlarged in the general case to include the Fredholm semigroups with respect to matrix algebras over A, embedded one in another in the natural way (in the upper left-hand corner, with 1 in the lower right-hand corner). Second, in the statement (ii), one should only consider cancellative abelian semigroups. (Then the alternative proof of Theorem 2(i), given in the last (parenthetical) paragraph, also establishes this modified statement of Theorem 2(ii) for cancellative abelian semigroups.)

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