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Lê Cycles and Milnor Classes of Compact Hypersurfaces

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Abstract. We determine the relation amongst the global Lê cycles and the Milnor classes of analytic hypersurfaces defined by a section of a very ample line bundle over a compact complex manifold. The key point is finding appropriate expressions for the global Lê cycles and for the Milnor classes in terms of polar varieties. Our starting points are an interpretation of the Lê cycles given by T. Gaffney and R. Gassler, a formula by A. Parusinski and P. Pragacz for the Milnor classes via McPherson's functor, and a conjecture of J.-P. Brasselet, that we prove, stating that Milnor classes can be expressed in terms of polar varieties. We then use the work by R. Piegne for Mather classes, by J. Schürmann and M. Tibăr for MacPherson's classes for constructible functions, and by D. Massey for an extension of the local Lê cycles for constructible sheaves.

Résumé. Nous déterminons la relation entre les cycles de Lê globaux et les classes de Milnor des hypersurfaces analytiques définies par une section d'un fibré en droites très ample sur des variétés non-singulières complexes compactes. Le point clé consiste à trouver des expressions appropriées des cycles de Lê globaux et des classes de Milnor en termes de variétés polaires. Nos points de départ sont une interprétation des cycles de Lê donnée par T. Gaffney et R. Gassler, une formule de A. Parusinski et P. Pragacz pour les classes de Milnor via le foncteur de McPherson, et une conjecture de J.-P. Brasselet pour les classes de Milnor, que nous démontrons, qui affirme que l'on peut exprimer les classes de Milnor en fonction des classes polaires. Nous utilisons alors des travaux de R. Piene sur les classes de Mather, de J. Schürmann et M. Tibăr sur les classes de MacPherson des fonctions constructibles, et de D. Massey qui généralise les cycles de Lê locaux aux faisceaux constructibles.

Introduction. Lê cycles were introduced by D. Massey in [14]. These are local invariants of holomorphic map-germs that determine the topology of the

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local Milnor fibre: If a map germ

$$(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$$

has a (possibly non-isolated) critical point at $\underline{0}$, then its local Milnor fibre F is an n-ball to which we attach handles of various dimensions, as described by the multiplicities of the corresponding Lê cycles. Using [10], the concept of Lê cycles extends naturally to global singular hypersurfaces in complex manifolds. These are elements in the Chow group, whose restriction to a neighborhood of each point are the classes of the usual Lê cycles. In other words, let Z be a hypersurface defined as the zero set of a section s of a holomorphic line bundle over M, let U_i be a local chart around a point in Z, and let f_i be the restriction $s|_{U_i}$; then each global Lê cycle Λ^k_s determines the class of the corresponding (local) Lê cycle $\Lambda^k_{f_i}$:

$$\Lambda^k_{s|_{U_i}} = [\Lambda^k_{f_i}].$$

There is a global Lê cycle in each dimension from 0 up to the dimension of the singular set.

Milnor classes are homology classes associated to compact complex hypersurfaces in complex manifolds. These are, by definition, the difference between the Schwartz–MacPherson and the Fulton–Johnson classes, each being a generalisation of (homology) Chern classes for singular varieties. Milnor classes were originally defined as elements in the singular homology group of the hypersurface, but they can actually be considered as cycles in the corresponding Chow group, by [9] and [11]. Just as Lê cycles, Milnor classes have support in the singular set of the variety, and there is a Milnor class in each complex dimension. So it is natural to ask what is the relation among the global Lê cycles and the Milnor classes.

Milnor classes were introduced in [1], [6], [7], [25], [18] and their name comes from the fact that when the singularities of Z are all isolated, the class in dimension 0, an integer, is the sum of the local Milnor numbers (by [23]), while the classes in the other dimensions are all 0 (by [24]).

There are several recent interesting articles about Milnor classes by various authors, as for instance P. Aluffi, J.-P. Brasselet, J. Schürmann, M. Tibăr, S. Yokura, T. Ohmoto and others. The Milnor classes also appear in [3] in relation with perturbative quantum field theory, and in [4] in relation with Donaldson–Thomas type invariants for Calabi–Yau threefolds. Even so, the rich geometric and topological information that the Milnor classes encode is not yet well understood.

The aim of this note is two-fold. On one hand we use Milnor classes to get global information about Lê cycles, and on the other hand we use Lê cycles to get geometric and topological information about Milnor classes. We prove the following theorem, which expresses each Milnor class of a hypersurface Z as a polynomial in the Lê cycles and the Chern class of its tautological bundle.

Theorem 1. *Let M be a compact complex variety with L a very ample line bundle on M. Consider the zero set Z of a holomorphic section s of L. Then the k-th Milnor class of Z, $\mathcal{M}_k(Z)$, is described as:*

$$\mathcal{M}_k(Z) = (i_{\text{sing}})_* \left(\sum_{j \geq 0} \sum_{i \geq k+j} (-1)^{i+j} \binom{i+1}{k+j+1} c_1(L|_{Z_{\text{sing}}})^{i-k} \cap \Lambda_s^i \right),$$

where $i_{\text{sing}} \colon Z_{\text{sing}} \hookrightarrow Z$ is the natural inclusion, the Λ_s^i are the Lê cycles of Z, and $c_1(L|_{Z_{\text{sing}}})$ is the Chern class of the corresponding line bundle.

Recall (see [15]) that at each point in Z the local Milnor fibre F is an n-ball to which we must attach handles of various dimensions, as described by the multiplicities of the Lê cycles. Hence this theorem suggests that the Milnor classes are determined by the vanishing homology of Z. This is the kernel of the specialization morphism $H_*(Z_t) \to H_*(Z)$, where Z_t is the complex manifold defined by the intersection of the zero section of the bundle L with a section near the one that determines Z, but which is everywhere transversal to the zero section of L. This gives a positive answer to a question raised in the introduction of [7] (*cf.* [20], [21]).

For the top degree $d = \dim Z_{\text{sing}}$ we prove the corollary below, which extends and strengthens [7, Corollary 5.13] in the hypersurface case.

Corollary. *Equip M with a Whitney stratification $\{Z_\beta\}$ adapted to Z. Then*

$$\mathcal{M}_d(Z) = \sum_{S_\beta \in Z_{\text{sing}}, \dim S_\beta = d} \mu^\perp(S_\beta)[\bar{S_\beta}] = \sum_{S_\beta \in Z_{\text{sing}}, \dim S_\beta = d} \lambda^d_{S_\beta}[\bar{S_\beta}],$$

where the sums run over the strata of dimension d that are contained in Z_{sing}, $\mu^\perp(S_\beta)$ is the transversal Milnor number of S_β, and $\lambda^d_{S_\beta}$ is the d-th Lê number of S_β. Furthermore, this equality is as cycles in the Chow group of Z and hence in the singular homology group of Z. Thus, up to sign, the class in singular homology represented by the Lê cycle Λ_s^d is the Milnor class $\mathcal{M}_d(Z)$.

One condition imposed in this paper is to assume that the bundle L is very ample. This was motivated by the description of R. Piene of the Mather classes of projective varieties, but various facts seem to suggest that this condition on L may not be necessary.

The trail for getting to Theorem 1 can be roughly described as follows. The first step is to define the global Lê cycles. This is done using the interpretation of the local Lê cycles given by Gaffney and Gassler in [10]: By blowing up the singular set of $Z \subset M$ and looking at the Chern class of the tautological bundle of the corresponding exceptional divisor. Next we observe that the main theorem of Parusinski and Pragacz in [18] expresses the total Milnor class as a function of the Schwartz–MacPherson classes of the closure of the strata of a Whitney stratification:

$$\mathcal{M}(Z) := \sum_{S \in \mathcal{S}} \gamma_S \big(c(L_{|Z})^{-1} \cap (i_{\bar{S},Z})_* c^{SM}(\bar{S}) \big).$$

On the other hand, Brasselet in [5] conjectured that the Milnor classes can be expressed in terms of polar varieties, which brings us closer to our goal of comparing Milnor classes with Lê cycles (which are defined via polar varieties in [14], [15]). Following this path, we notice that in [22], in the affine context, the Schwartz–MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ are described using algebraic cycles in the Chow groups, which they call MacPherson cycles. We prove the analogous result in the projective case. In this construction a key tool is played by certain projective polar varieties. This suffices for us, since the assumption of considering a very ample line bundle L over M implies that M is a projective variety. We prove (see [8] for details and explanations):

THEOREM 2. *Let X be a projective variety. Consider $X^n \overset{\varphi}{\hookrightarrow} \mathbb{C}P^N$ a closed immersion and $\mathcal{L} = \mathcal{O}_{\mathbb{C}P^n}(1)$. Let $\{S_\alpha\}$ be a Whitney stratification of X with $\overline{S}_\alpha \overset{i_{\overline{S}_\alpha,X}}{\hookrightarrow} X$. Then the k-th Schwartz–MacPherson class of X, $c_k^{SM}(X)$, is given by*

$$\sum_\alpha \eta(S_\alpha, 1_X) \sum_{i=k}^{d_\alpha} (-1)^{d_\alpha - i} \binom{i+1}{k+1} (i_{\overline{S}_\alpha,X})_* \big(c_1(\varphi^* \mathcal{L})^{i-k} \cap [\mathbb{P}_i(\overline{S_\alpha})] \big),$$

where $\eta(S_\alpha, 1_X)$ is the normal Morse index.

A key point for proving Theorem 2 is Piene's characterization of the Mather classes via polar varieties [19, Théorème 3].

Due to the construction of the Schwartz–MacPherson classes via the Chern–Mather classes [13], the difference between Schwartz–MacPherson and Chern–Mather classes is supported on the singular locus. Hence these two classes are the same on the degrees higher than the dimension of the singular locus; for example, in the case of isolated singularities, these classes are identical in dimensions greater than zero (*cf.* [17]).

We remark that there is another formula for the MacPherson classes in terms of polar varieties given in [12]. Yet, the expression we need for proving Theorem 1 is the one given by Theorem 2, because this allows comparison with the Lê cycles.

The final ingredient we need for proving Theorem 1 is Massey's Lê cycles for constructible sheaves via polar varieties. We extend these to the projective setting and prove a formula comparing them with the MacPherson cycles, analogous to Massey's formula in [16].

Theorem 1 is proved by considering the formula for the Milnor classes by Parusinski–Pragacz, mentioned above, replacing in it the Schwartz–MacPherson classes of X by the expression given in Theorem 2, and then using Massey's characterization of Lê cycles for constructible sheaves via polar varieties, extended to the projective setting.

The details and complete proofs of the results stated above will be given in [8].

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References

1. P. Aluffi, *Chern classes for singular hypersurfaces*. Trans. Amer. Math. Soc. **351** (1999), 3989–4026.

2. ______, *Weighted Chern–Mather classes and Milnor classes of hypersurfaces*. Adv. Stud. Pure Math. **29** (2000), 1–20.

3. P. Aluffi and M. Marcolli, *Feynman motives of banana graphs*. Commun. Number Theory Phys. **3** (2009), 1–57.

4. K. Behrend, *Donaldson–Thomas type invariants via microlocal geometry*. Ann. Math. (2) **170** (2009), 1307–1338.

5. J.-P. Brasselet, *Milnor classes via polar varieties*. Contemp. Math. **266** (2000), 181–187.

6. J.-P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor numbers and classes of local complete intersections*. Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), 179–183.

7. ______, *Milnor classes of local complete intersections*. Trans. Amer. Math. Soc. **354** (2001), 1351–1371.

8. R. Callejas-Bedregal, M. F. Z. Morgado and J. Seade, *Lê cycles and Milnor classes*. Preprint, 2010.

9. W. Fulton and K. Johnson, *Canonical classes on singular varieties*. Manuscripta Math. **32** (1980), 381–389.

10. T. Gaffney and R. Gassler, *Segre numbers and hypersurface singularities*. J. Algebraic Geometric **8** (1999), 695–736.

11. G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*. Comm. Algebra **18** (1990), 2821–2839.

12. D. T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulieres*. Ann. Math. (2) **114** (1981), 457–491.

13. R. MacPherson, *Chern classes for singular varieties*. Ann. of Math. **100** (1974), 423–432.

14. D. B. Massey, *The Lê varieties. I*. Invent. Math. **99** (1990), 357–376.

15. ______, *Lê Cycles and Hypersurface Singularities*. Lecture Notes in Mathematics **1615**, Springer-Verlag, Berlin, 1995.

16. ______, *Numerical invariants of perverse sheaves*. Duke Math. J. **73** (1994), 307–370.

17. T. Ohmoto, T. Suwa and S. Yokura, *A remark on the Chern classes of local complete intersections*. Proc. Japan Acad. **73** (1997), 93–95.

18. A. Parusiński and P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*. J. Algebraic Geom. **10** (2001), 63–79.

19. R. Piene, *Cycles polaires et classes de Chern pour les variétés projectives singulieres*. Travaux en Cours **37** (1988), 7–34.

20. J. Schürmann, *Lectures on characteristic classes of constructible functions*. Notes by Piotr Pragacz and Andrzej Weber. Trends Math., Topics in cohomological studies of algebraic varieties, Birkhäuser, Basel, 2005, 175–201.

21. ______, *Nearby cycles and characteristic classes of singular spaces*. ArXiv: 1003.2343v2[math.AG] 4 may 2010.

22. J. Schürmann and M. Tibăr, *MacPherson cycles and polar varieties of affine algebraic spaces*. Tohoku Math. J. **62** (2010), 29–44.

23. J. Seade and T. Suwa, *An adjunction formula for local complete intersections*. Internat. J. Math. **9** (1998), 759–768.

24. T. Suwa, *Classes de Chern des intersections complètes locales*. C. R. Acad. Sci. Paris **324** (1996), 67–70.

25. S. Yokura, *On characteristic classes of complete intersections*. Contemp. Math. **241** (1999), 349–369.

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