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On the Pickands Stochastic Process

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Abstract. We consider the Pickands process

$$P_n(s) = \log(1/s)^{-1} \log \frac{X_{n-k+1,n} - X_{n-[k/s]+1,n}}{X_{n-[k/s]+1,n} - X_{n-[k/s^2]+1,n}}, \qquad \Big(\frac{k}{n} \le s^2 \le 1\Big), \tag{1}$$

which is a generalization of the classical Pickands estimate $P_n(1/2)$ of the extremal index. We undertake here a purely stochastic process view for the asymptotic theory of that process by using the Csörgő–Csörgő–Horvàth–Mason [2] weighted approximation of the empirical and quantile processes to suitable Brownian bridges. This leads to the uniform convergence of the margins of this process to the extremal index and a complete theory of weak convergence of P_n in $\ell^\infty([a,b])$ to some Gaussian process

$$\{\mathbb{G}, a \le s \le b\} \tag{2}$$

for all $[a,b] \subset]0,1[$. This frame greatly simplifies the former results and enable applications based on stochastic processes methods.

Résumé. Nous considérons le processus de Pickands défini en (1) qui est une généralisation de l'estimateur classique de Pickands $P_n(1/2)$ de l'indice extremal. Nous abordons l'étude de ce processus du point de vue des processus stochastiques en établissant son comportement asymptotique. Nous utilisons comme outil principal l'approximation simultanée du processus empirique et du processus des quantiles uniformes dûe à Csörgő–Csörgő–Horvàth–Mason [2]. Nous établissons la convergence vague et uniforme du processus (1) vers un processus gaussien (2) entièrement décrit. Cette approche simplifie les résultats antérieurs et permet des applications basées sur des méthodes de processus stochastiques.

1. Introduction.

Let $X_1, \dots, X_n$ be a sequence of independant random variables of common distribution function F with $F(1) = 0$, and let $X_{1,n} < X_{2,n} < \cdots < X_{n,n}$, denote the order statistics of $X_1, \dots, X_n$ for any fixed $n \ge 1$. Let $k = k(n)$ be a sequence of positive integers satisfying:

$$k \to +\infty, \quad k/n \to 0, \quad \text{and} \quad \log\log n/k \to 0 \quad \text{as } n \to +\infty. \tag{K}$$

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We are concerned with the following stochastic process

$$P_n(s) = \log(1/s)^{-1} \log \frac{X_{n-k+1,n} - X_{n-[k/s]+1,n}}{X_{n-[k/s]+1,n} - X_{n-[k/s^2]+1,n}}, \quad \text{for } s^2 \geq \frac{k}{n}.$$

If $s^2 < \frac{k}{n}$, we put $P_n(s) = 0$. For $s = 1/2$, $P_n(1/2)$ is the so-called Pickands estimator of the extremal index γ when F is in the extremal domain of a Generalized Extreme Value Distribution

$$G_\gamma(x) = \exp\bigl(-(1+\gamma x)^{-\frac{1}{\gamma}}\bigr), \quad \text{for } 1+\gamma x > 0, \quad \text{with } \gamma \in \mathbb{R}.$$

This stochastic process has been studied by many authors: Alves [1], Falk [9], Pereira [12], Yun [16] and [17], Segers [14], *etc.* The latter gave a summary of the previous works.

The handling of the Pickands process heavily depends on that of the stochastic process of the large quantiles

$$\{X_{n-[k/s]+1,n}, n \geq 1\}. \tag{1.1}$$

Drees [8] and de Haan and Ferreira [6] provided asymptotic uniform and Gaussian approximation of (1.1) when F is in the extremal domain under second order conditions. Segers [14] used such results to construct new estimators of the extremal index with integrals of statistics on the form of $P_n(s)$. This motivates us to undertake a purely stochastic process approach of the Pickands process in a simpler way but in a more adequate handling in order to derive from this study many potential applications.

2. Results.

Let us introduce some notation. First define the generalized inverse of F:

$$F^{-1}(s) = \inf\{x, F(x) \geq s\}, \quad 0 < s < 1,$$

and let

$$p_n(s) = \log(1/s)^{-1} \log \frac{F^{-1}(1-[k/s]/n) - F^{-1}(1-k/n)}{F^{-1}(1-[k/s^2]/n) - F^{-1}(1-[k/s]/n)}.$$

We are going to investigate this Pickands process

$$\Bigl\{\kappa_n(s), \frac{k}{n} < s^2 < 1\Bigr\} = \Bigl\{\sqrt{k}\bigl(P_n(s) - p_n(s)\bigr), \frac{k}{n} < s^2 < 1\Bigr\}.$$

But it is easy to see that $P_n(s) \to K(\gamma) = \gamma \mathbf{1}_{\{\gamma \neq +\infty\}}$ for $F \in D(G_{1/\gamma})$. This extends the motivation to the study of

$$\Bigl\{\kappa_n^*(s), \frac{k}{n} < s^2 < 1\Bigr\} = \Bigl\{\sqrt{k}\bigl(P_n(s) - K(\gamma)\bigr), \frac{k}{n} < s^2 < 1\Bigr\}.$$

Since our conditions depend on auxiliary functions of the representations of functions F in the extremal domain, that is $F \in D(G_{1/\gamma})$, $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$, we feel obliged to introduce them. First, for the Gumbel case $\gamma = +\infty$, we have

$$F^{-1}(1-u) = d - s(u) + \int_u^1 t^{-1} s(t)\, dt, \tag{2.1}$$

where

$$s(u) = c\big(1 + p(u)\big) \exp\left(\int_u^1 t^{-1} b(t)\, dt\right), \quad 0 < u < 1,$$

is a slowly varying function in the neighborhood of 0. For the Frechet case $\gamma > 0$, we have

$$F^{-1}(1-u) = c\big(1 + p(u)\big) u^{-K(\gamma)} \exp\left(\int_u^1 t^{-1} b(t)\, dt\right). \tag{2.2}$$

For the Weibull case $\gamma < 0$, it is known that $x_0(F) = \sup\{x, F(x) < 1\} < \infty$, and we have

$$x_0 - F^{-1}(1-u) = c\big(1 + p(u)\big) u^{K(\gamma)} \exp\left(\int_u^1 t^{-1} b(t)\, dt\right). \tag{2.3}$$

In all these cases, $\big(p(u), b(u)\big) \to (0,0)$ when $u \to 0$ and c is a positive constant. The representations (2.2) and (2.3) are the Karamata representation, while (2.1) is that of de Haan [5].

We fix two numbers a and b, $a < b$, such that $[a,b] \subset]0,1[$. For each $\gamma > 0$, we will denote $a(u) = F^{-1}(1-u)$ and for $\gamma < 0$, $a(u) = x_0 - F^{-1}(1-u)$. Set

$$k_{a^2} = [a^{-2}]k \tag{2.4}$$

and $\lambda > 1$ a real number. Finally we define for an arbitrary function h defined on $(0,1)$ in $\mathbb{R}$,

$$h_n(\lambda, h, a) = \sup_{0 \le t \le \lambda k_{a^2}/n} |h(t)|.$$

We shall consider the regularity conditions

$$\sqrt{k}\, p_n(\lambda, p, a) \to 0, \quad \text{as } n \to +\infty. \tag{RC1}$$

and

$$\sqrt{k}\, b_n(\lambda, b, a) \to 0, \quad \text{as } n \to +\infty. \tag{RC2}$$

All unspecified limits occur when $n \to \infty$. Finally we denote by $o_p(s,a)$ and $o_p(s,a,b)$, respectively the uniform limits in $s \in [0,1]$ and $s \in [a,b]$. Here are our main results.

THEOREM 1. *Let $F \in D(G_{1/\gamma})$, $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$. Let $0 < a < b < 1$. If* (RC1) *and* (RC2) *hold, then $\{\kappa_n(s), s \in [a,b]\}$ converges to a Gaussian process $\{\mathbb{G}(s), a < s < b\}$ in $\ell^\infty([a,b])$, of covariance function*

(2.5)

$$\Gamma(s,t) = \frac{1}{(s^{-K(\gamma)}-1)(t^{-K(\gamma)}-1)\log s \log t}$$
$$\times \{(s^{-K(\gamma)}-1)[t(t^{-K(\gamma)}-1) + t^{-K(\gamma)}K(\gamma)s - K(\gamma)t^2 - K(\gamma)^2t^2]$$
$$+ (t^{-K(\gamma)}-1)[K(\gamma)(s^{-K(\gamma)}t - s^2)] + K(\gamma)^2 t^{-K(\gamma)}[s^{-K(\gamma)} - s^2]\},$$

with the convention that for $K(\gamma) = 0$,

$$\Gamma(s,t) = \frac{1-s^2}{(\log s)^2(\log t)^2}.$$

Also $\{\kappa_n^*(s), a < s < b\}$ converges to $\mathbb{G}$, in $\ell^\infty([a,b])$.

3. **Proof of the results.** It is based on the so-called Hungarian construction of Csörgő *et al.* [2]. For this define by $\{U_n(s), 0 \le s \le 1\}$, the uniform empirical distribution function and $\{V_n(s),\ 0 \le s \le 1\}$, the uniform empirical quantile function, based on the $n \ge 1$ first observations $U_1, U_2, \ldots$ sampled from a uniform random variables on $(0,1)$ and let

$$\{\beta_n(s); 0 \le s \le 1\} = \left\{\sqrt{n}\big(U_n(s) - s\big), 0 \le s \le 1\right\}$$

be the corresponding empirical process. One version of [2] is the following.

THEOREM 2 ([2]). *There exists a probability space holding a sequence of independant random variables $U_1, U_2, \ldots$ uniformly distributed on $(0,1)$ and a sequence of Brownian bridges*

$$\{B_n(t), 0 \le t \le 1\} = \{W_n(t) - tW_n(1), 0 \le t \le 1\},$$

where the W_n are Wiener processes, such that for any $0 < \nu < 1/2$ and $d \ge 0$,

$$\sup_{d/n \le s \le 1-d/n} \frac{|\beta_n(s) - B_n(s)|}{\{s(1-s)\}^{1/2-\nu}} = O_p(n^{-\nu}).$$

From this, we derive the Gaussian approximation of the process

$$\left\{U_{[k/s],n} - \frac{[k/s]}{n}, \frac{k}{n} \le s \le 1\right\}$$

in the following lemma.

Lemma 1. *On the probability space in* [2] *we have*

$$\sup_{k/(n-1)\le s\le 1}\Big|\sqrt{k}\Big(\frac{n}{[k/s]}U_{[k/s],n}-1\Big)-W_n(1,s)\Big| = O_p(1),$$

where $W_n(1,s)=s(\frac{k}{n})^{-1/2}W_n(k/ns)$ *is a Wiener process.*

Our proof is performed on the space of Theorem 2. For conciseness, we restrict ourselves to the case $\gamma>0$, since the other cases are proved similarly.

Let us begin by establishing that

$$\frac{\sqrt{k}\{X_{n-[k/s]+1,n}-F^{-1}(1-\frac{[k/s]}{n})\}}{a(k/n)} = -K(\gamma)s^{K(\gamma)}W_n(1,s)+o_p(s,a,b). \tag{3.1}$$

Let

$$A_n(s)=X_{n-k+1,n}-X_{n-[k/s]+1,n},\quad B_n(s)=X_{n-[k/s]+1,n}-X_{n-[k/s^2]+1,n}$$

and

$$a_n(s)=F^{-1}\Big(1-\frac{k}{n}\Big)-F^{-1}\Big(1-\frac{[k/s]}{n}\Big),$$
$$b_n(s)=F^{-1}\Big(1-\frac{[k/s]}{n}\Big)-F^{-1}\Big(1-\frac{[k/s^2]}{n}\Big).$$

By using (2.2), we have

$$\frac{X_{n-[k/s]+1,n}}{F^{-1}(1-\frac{[k/s]}{n})}=\frac{1+p(U_{[k/s],n})}{1+p([k/s]/n)}\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-K(\gamma)}\exp\Big(\int_{U_{[k/s],n}}^{[k/s]/n}t^{-1}b(t)\,dt\Big).$$

We shall treat the three items one by one. Consider $s\in[a,1]$, $a>0$. Then $[k/s]\le[a^{-1}]k=k_a$. Since $\frac{n}{k_a}U_{k_a,n}\to 1$ in probability, for each $\varepsilon>0$ and for each $\lambda>1$, we have, with probability at least greater than $1-\varepsilon$, for large values of n,

$$U_{k_a,n}\le\frac{\lambda k_a}{n}.$$

Since (RC1) holds,

$$\sqrt{k}p_n(\lambda,p,a)=\sqrt{k}\sup_{0<u<\lambda k_a/n}|p(u)|\to 0,$$

and then

$$\sqrt{k}\Big(\frac{1+p(U_{[k/s],n})}{1+p([k/s]/n)}-1\Big)\to 0 \tag{3.2}$$

in probability, uniformly in $s \in [a,b]$. Then, using the mean value theorem,

$$\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-K(\gamma)} - 1 = -K(\gamma)\Big(\frac{n}{[k/s]}U_{[k/s],n} - 1\Big)b_n(s),$$

with

$$b_n(s) \in \Big[\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big) \wedge 1, \Big(\frac{n}{[k/s]}U_{[k/s],n}\Big) \vee 1\Big]. \tag{3.3}$$

Now by Lemma 1, uniformly in $s \in [a,b]$,

$$\sqrt{k}\Big(\frac{n}{[k/s]}U_{[k/s],n} - 1\Big) = W_n(1,s) + O_p(1).$$

Since $\sup_s |W_n(1,s)|$ is a finite random variable in probability,

$$\frac{n}{[k/s]}U_{[k/s],n} \to 1$$

uniformly in probability and therefore $b_n(s) \to 1$ uniformly in probability. Let us set $o_p(s,a,b)$ as meaning: tends to zero uniformly in $s \in [a,b]$. We have

$$\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-K(\gamma)} - 1 = -K(\gamma)\Big(\frac{n}{[k/s]}U_{[k/s],n} - 1\Big)\big(1 + o_p(s,a,b)\big).$$

It follows by applying Lemma 1 that

$$\sqrt{k}\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-K(\gamma)} - 1 = -K(\gamma)W_n(1,s) + o_p(s,a,b).$$

By the same techniques, we easily get for $\varepsilon > 0$, with probability at least greater than $1-\varepsilon$,

$$\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-b_n(\lambda,a,b)} \le \exp\Big(\int_{U_{[k/s],n}}^{[k/s]/n} t^{-1}b(t)\,dt\Big) \le \Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{b_n(\lambda,a,b)}, \tag{3.4}$$

for large values of n. We must show that both extreme terms tend to unity uniformly in $s \in [a,b]$. Indeed

$$\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{b_n(\lambda,a,b)} - 1 = b_n(\lambda,a,b)\Big(\frac{n}{[k/s]}U_{[k/s],n} - 1\Big)b_n(s),$$

where $b_n(s)$ is already defined in (3.3) and is $o_p(s,a,b)$. We arrive at

$$\sqrt{k}\Big\{\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{b_n(\lambda,a,b)} - 1\Big\}$$
$$= b_n(\lambda,a,b)\big(W_n(1,s) + o_p(s,a,b)\big)\big(1 + o_p(s,a,b)\big)$$

which is an $o_p(s,a,b)$ term, since $\sup_{s\in(a,b)}|W_n(1,s)|$ is bounded in probability. Further, put

$$A_{1,n}(s)=\frac{1+p(U_{[k/s],n})}{1+p([k/s]/n)},\quad A_{2,n}(s)=\Big(\frac{n}{[k/s]}U_{[k/s],n}\Big)^{-K(\gamma)}$$

and

$$A_{3,n}(s)=\exp\bigg(\int_{U_{[k/s],n}}^{[k/s]/n}t^{-1}b(t)\,dt\bigg).$$

Now we have

$$\frac{X_{n-[k/s]+1,n}}{F^{-1}(1-\frac{[k/s]}{n})}-1=A_{2,n}(s)A_{3,n}(s)\{A_{1,n}-1\}+A_{2,n}\{A_{3,n}-1\}+\{A_{2,n}-1\}.$$

This leads to

$$\begin{aligned}&\sqrt{k}A_{2,n}(s)A_{3,n}(s)\{A_{1,n}-1\}+A_{2,n}\{A_{3,n}-1\}+\{A_{2,n}-1\}\\&\quad=A_{2,n}(s)A_{3,n}(s)\sqrt{k}\{A_{1,n}-1\}+A_{2,n}\sqrt{k}\{A_{3,n}-1\}+\sqrt{k}\{A_{2,n}-1\}.\end{aligned}$$

Finally, we arrive at

$$\frac{\sqrt{k}\{X_{n-[k/s]+1,n}-F^{-1}(1-\frac{[k/s]}{n})\}}{a([k/s]/n)}=-K(\gamma)W_n(1,s)+o_p(s,a,b),$$

where, for $a(s)=F^{-1}(1-s)$, we used the result that $a([k/s]/n)/a(k/n)$ tends uniformly to $s^{-K(\gamma)}$ for $s\in(a,b)$. This achieves the proof of (3.1).

Now we use (3.1) to prove Theorem 1. Put

$$W_n(2,s)=\Big(\frac{k}{n}\Big)^{-1/2}W_n(k/n),\quad W_n(3,s)=s^2\Big(\frac{k}{n}\Big)^{-1/2}W_n(s^{-2}k/n).$$

This latter is a Gaussian process with covariance function $\min(s^2,t^2)$. Denote now

$$C_n(s)=\frac{A_n(s)}{B_n(s)}=\frac{X_{n-[k/s]+1,n}-X_{n-k+1,n}}{X_{n-[k/s^2]+1,n}-X_{n-[k/s]+1,n}}$$

and

$$c_n(s)=\frac{a_n(s)}{b_n(s)}=\frac{F^{-1}(1-\frac{[k/s]}{n})-F^{-1}(1-\frac{k}{n})}{F^{-1}(1-\frac{[k/s^2]}{n})-F^{-1}(1-\frac{[k/s]}{n})}.$$

Then

$$\log C_n(s)-\log c_n(s)=\big(C_n(s)-c_n(s)\big)\times c_n(1,s)^{-1},$$

where

$$c_n(1,s)\in[C_n(s)\wedge c_n(s),C_n(s)\vee c_n(s)]\to s^{-K(\gamma)},$$

uniformly in $s \in (0,1)$ and $c_n(1,s)^{-1} \to s^{K(\gamma)}$ uniformly in $s \in [a,1]$. Then

$$\log C_n(s) - \log c_n(s) = \big(s^{K(\gamma)} + o_p(s,a)\big)\big(C_n(s) - c_n(s)\big).$$

Next

$$C_n(s) - c_n(s) = \frac{-a_n(s)\big(B_n(s) - b_n(s)\big) + b_n(s)\big(A_n(s) - a_n(s)\big)}{b_n(s)B_n(s)}.$$

The same technique leads to

$$a_n(s) = a\big([k/s]/n\big)(1 - s^{-K(\gamma)})\big(1 + o_p(s,a)\big),$$
$$A_n(s) = a(U_{[k/s],n})(1 - s^{-K(\gamma)})\big(1 + o_p(s,a)\big)$$

and

$$b_n(s) = a([k/s^2]/n)(1 - s^{-K(\gamma)})\big(1 + o_p(s,a)\big),$$
$$B_n(s) = a(U_{[k/s^2],n})(1 - s^{-K(\gamma)})\big(1 + o_p(s,a)\big).$$

Next

$$B_n(s) - b_n(s) = a([k/s^2]/n)\Big(\frac{X_{n-[k/s^2]+1,n}}{F^{-1}(1 - [k/s^2]/n)} - 1\Big)$$
$$- a([k/s]/n)\Big(\frac{X_{n-[k/s]+1,n}}{F^{-1}(1 - [k/s]/n)} - 1\Big).$$

When gathered, these formulas imply

$$\sqrt{k}\big(B_n(s) - b_n(s)\big) = a([k/s^2]/n)\big(1 + o_p(s,a)\big)\big(-K(\gamma)W_n(3,s) + o_p(s,a)\big)$$
$$- a([k/s]/n)\big(1 + o_p(s,a)\big)\big(-K(\gamma)W_n(1,s) + o_p(s,a)\big)$$

and

$$\sqrt{k}\frac{-a_n(s)(B_n(s) - b_n(s))}{b_n(s)B_n(s)} = -K(\gamma)\frac{s^{-K(\gamma)}}{(1 - s^{-K(\gamma)})}W_n(3,s)$$
$$+ K(\gamma)\frac{s^{-2K(\gamma)}}{(1 - s^{-K(\gamma)})}W_n(1,s) + o_p(s,a).$$

Similarly, we have

$$\sqrt{k}\frac{b_n(s)\big(A_n(s) - a_n(s)\big)}{b_n(s)B_n(s)} = -K(\gamma)\frac{s^{-K(\gamma)}}{(1 - s^{-K(\gamma)})}W_n(1,s)$$
$$+ K(\gamma)\frac{s^{-2K(\gamma)}}{(1 - s^{-K(\gamma)})}W_n(2,s) + o_p(s,a).$$

We conclude that

$$\begin{aligned}&\big(\log(1/s)\big)^{-1}\sqrt{k}\{\log C_n(s)-\log c_n(s)\}\\&\quad=\log(1/s)^{-1}\frac{K(\gamma)}{(1-s^{-K(\gamma)})}\big(1+o_p(s,a)\big)\\&\qquad\times\big\{(s^{-K(\gamma)}-1)\big(W_n(1,s)+o_p(s,a)\big)\\&\qquad\quad+s^{-K(\gamma)}\big(W_n(2,s)+o_p(s,a)\big)-W_n(3,s)+o_p(s,a)\big\}.\end{aligned}$$

Now, by restricting ourselves to $s\in[a,b]\subset\,]0,1[$, we get, uniformly in those s,

$$\kappa_n(s)=\sqrt{k}\{P_n(s)-\log c_n(s)/\log(1/s)\}=\mathbb{G}_n(s)+o_p(s,a,b),$$

where

$$\mathbb{G}_n(s)=\frac{K(\gamma)}{(s^{-K(\gamma)}-1)\log s}\big\{(s^{-K(\gamma)}-1)W_n(1,s)+s^{-K(\gamma)}W_n(2,s)-W_n(3,s)\big\}$$

is a Gaussian process with covariance $\Gamma_n(s,t)\to\Gamma(s,t)$ expressed in Theorem 1. To extend our result to $\kappa_n^*(s)$ we have to prove that under (RC1) and (RC2),

$$\sqrt{k}\big(p_n(s)-K(\gamma)\big)\to 0. \tag{3.5}$$

In Theorem 1, the asymptotic laws comes from that of $\frac{n}{k}U_{[k/s],n}$. All the remainder terms are controlled uniformly by the regularity conditions. If we treat (3.5) the corresponding part $\frac{n}{k}[k/s]/n$ satisfies:

$$\sqrt{k}\left(\frac{n}{k}[k/s]/n-\left(\frac{1}{s}\right)^{K(\gamma)}\right)\to 0$$

uniformly in $s\in[a,b]$. By respectively the same proofs, we arrive at the result for $\gamma<0$, $\gamma>0$, $\gamma=+\infty$, and this completes the proof.

4. **Applications.** Consider a measure m bounded on compacts set $[a,b]\subset\,]0,1[$ and (as Segers [14]) define the statistics

$$\mathbb{I}_n(a,b,m)=\int_a^b P_n(s)\,dm(s).$$

with $K(\gamma)=\gamma 1_{\gamma\neq+\infty}$. We get

$$\begin{aligned}\sqrt{k}\big(\mathbb{I}_n(a,b,m)-K(\gamma)\big)&=\int_a^b\sqrt{k}\big(P_n(s)-K(\gamma)\big)\,dm(s)\\&=\int_a^b\mathbb{G}_n(s)\,dm(s)\to\int_a^b\mathbb{G}(s)\,dm(s)=\mathbb{I}(a,b,m).\end{aligned}$$

THEOREM 3. *Let $0 < a < b < 1$, and let m be a measure bounded on compact set $[a,b] \subset]0,1[$. Let $F \in D(G_{1/\gamma})$, $\gamma < 0$, $\gamma > 0$, $\gamma = +\infty$, and assume* (RC1) *and* (RC2) *holds. Then*

$$\sqrt{k}\big(\mathbb{I}_n(a,b,m) - K(\gamma)\big) \to \mathcal{N}(0, \sigma_m^2)$$

with

$$\sigma_m^2 = \mathbb{E}\int_a^b \mathbb{G} + (s)\, dm(s) = \int_a^b \int_a^b \Gamma(s,t)\, dm(s)\, dm(t).$$

We can investigate the problem of minimizing σ_m^2, that is finding

$$\min_{m \in \mathcal{M}} \int_a^b \int_a^b \Gamma(s,t)\, dm(s)\, dm(t),$$

in future works. Recall that for $m_s = \delta_s$, we have $\mathbb{I}_n(a,b,m_s) = P_n(s)$.

5. Continuity modulus.

LEMMA 2. *We have, almost surely, for any $s \in [a,b] \subset]0,1[$*

$$\lim_{h\to 0} \sup_{|s-t|\leq h} \frac{|\kappa_n(s) - \kappa_n(t)|}{w(h)} \leq L(\gamma)$$

with

$$L(\gamma) = \frac{K(\gamma)}{|\log b|} + \frac{K(\gamma)}{(|b^{-K(\gamma)} - 1|\,|\log b|)},$$

with the convention that

$$L(\gamma) = \frac{1}{(|\log b|)^2} \quad \textit{for } K(\gamma) = 0$$

and $w(h) = \sqrt{2h\log(1/h)}$.

In other words, the Gaussian process $\kappa_n(s)$ has modulus of continuity $w_\kappa\big(\sqrt{2\delta\log(1/\delta)}\big)$ with probability 1 for sufficiently small $\delta > 0$.

PROOF. Put

$$g(s) = \frac{K(\gamma)}{(s^{-K(\gamma)} - 1)\log s}$$

and

$$h_n(s) = (s^{-K(\gamma)} - 1)W_n(1,s) + s^{-K(\gamma)}W_n(2,s) - W_n(3,s).$$

We have the decomposition

$$|\mathbb{G}_n(s) - \mathbb{G}_n(t)| = \big|\big(g(s) - g(t)\big)h_n(s)\big| + \big|g(t)\big(h_n(s) - h_n(t)\big)\big|.$$

Now by using the exact continuity modulus of the Wiener Process $W_n(\,\cdot\,,s)$, and by repeatingly using the mean value theorem, we easily arrive at the results. We emphasize that these results hold for a compact set $[a,b]$, $0 < a < b < 1$. □

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References

1. M. I. F. Alves, *Estimation of the tail parameter in the domain of attraction of an extremal distribution.* J. Statist. Plann. Inference **45** (1995), 143–173.
2. M. Csörgő, S. Csörgő, L. Horvàth, and M. Mason, *Weighted empirical and quantile processes.* Ann. Probab. **14** (1986), 31–85.
3. M. Csörgő and P. Revesz, *Strong Approximations in Probability and Statistics.* Academic Press, Budapest, 1981.
4. S. Csörgő, P. Deheuvels and D. M. Mason, *Kernel estimates of the tail index of a distribution.* Ann. Statist. **13** (1985), 1050–1077.
5. L. de Haan, *On regular variation and its application to the weak convergence of sample extremes.* Mathematical Centre Tracts **32**, Mathematisch Centrum, Amsterdam, 1970.
6. L. de Haan and A. Ferreira, *Extreme value theory: An introduction.* Springer, New York, 2006.
7. L. de Haan and S. I. Resnick, *A simple asymptotic estimate for the index of a stable law.* J. Roy. Statist. Soc. **42** (1980), 83–87.
8. H. Drees, *Refined Pickands Estimators for the extreme value index.* Ann. Statist. **23** (1995), 2059–2080.
9. M. Falk, *Efficiency of convex combinations of Pickands' estimator of the extreme value index.* J. Nonparametric Statist. **4** (1994), 133–147.
10. J. Galambos, *The Asymptotic theory of Extreme Order Statistics.* Wiley, New York, 1985.
11. E. Hauesler and L. Teugels, *On asymptotic normality of Hill's estimator for the exponent of regular variation.* Ann. Statist. **13** (1985), 743–756.
12. T. T. Pereira, *Second order behavior of domains of attraction and the bias of generalized Pickands' estimator.* In: Extreme Value Theory and Applications III (J. Galambos, J. Lechnar and E. Simiu, eds.), NIST, 1994, 165–177.
13. S. I. Resnick, *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.* Springer, New York, 2006.
14. J. Segers, *Generalized Pickands Estimators for the Extreme Value Index* J. Statist. Plann. Inference 128, **2** (2002), 381–396.
15. A. W. van der Vaart and J. A. Wellner, *Weak Convergence and Empirical Processes With Applications to Statistics.* Springer, New York, 1996.
16. S. Yun, *A class of Pickands-type estimator for the extreme value index.* J. Statist. Plann. Inference **83** (2000), 113–124.
17. ———, *On a generalized Pickands estimator of the extreme value index.* J. Statist. Plann. Inference **102** (2002), 389–409.

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