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A Counterexample to Durfee's Conjecture

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Abstract. An old conjecture of Durfee [7] bounds the ratio of two basic invariants of complex isolated complete intersection surface singularities: the Milnor number and the singularity (or geometric) genus. We give a counterexample for the case of non-hypersurface complete intersections, and we formulate a weaker conjecture valid in arbitrary dimension and codimension. This weaker bound is asymptotically sharp. In this note we support the validity of the new proposed inequality by its verification in certain (homogeneous) cases. In a subsequent paper we will prove it for several other cases and we will provide a more comprehensive discussion.

Résumé. Une conjecture vieux de Durfee [7] limite le ratio de deux invariants fondamentales des singularités des surfaces complexes qui sont intersections complètes: le nombre de Milnor et la genre (géométrique) de la singularité. Nous presentons un contre-exemple pour le cas des intersections complètes (non-hypersurface), et nous formulons une conjecture plus faible en dimensions et codimensions arbitraires. Cette bond est forte asymptotiquement. Dans cette note nous appuyons la validité de l'inégalité nouvelle par sa vérification dans certains cases homogènes. Dans le papier prochaine nous allons le prouver pour plusieurs cases et nous fournirons une analyse plus complète.

1. **Introduction.** Let $(X,0)\subset(\mathbb{C}^N,0)$ be the analytic germ of a complex isolated complete intersection singularity (ICIS). Among various singularity invariants the two most basic ones are:

- the Milnor number μ, which measures the change of the local topological/homological Euler characteristic in smoothing deformation;
- the singularity genus (called also geometric genus) p_g, which measures the change of the local analytic Euler characteristic (Todd index) when we replace $(X,0)$ by its resolution.

Their definitions and some of their properties are given in Section 2.

The relation of these two invariants was investigated intensively, giving rise to several open problems as well. In particular, in [7] two conjectures were formulated:

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- *strong inequality*: if $(X,0)$ is an isolated complete intersection surface singularity, then $\mu \geq 6p_g$;
- *weak inequality*: if $(X,0)$ is a normal surface singularity (not necessarily ICIS) which admits a smoothing with Milnor number (second Betti number of the fiber) μ, then $\mu + \mu_0 \geq 4p_g$ (where μ_0 is the rank of the kernel of the intersection form).

Quite soon a counterexample to the weak conjecture was given in [27, p. 240] providing a normal surface singularity (not ICIS) with $\mu = 3$, $\mu_0 = 0$ and $p_g = 1$.

On the other hand, the 'strong inequality' valid for an ICIS was believed to be true (though hard to prove) and was verified in many particular hypersurfaces. For example,

- [26] proved $8p_g < \mu$ for $(X,0)$ of multiplicity 2,
- [4] proved $6p_g \leq \mu - 2$ for $(X,0)$ of multiplicity 3,
- [28] proved $6p_g \leq \mu - \text{mult}(X,0) + 1$ for quasi-homogeneous singularities,
- [21], [22] proved $6p_g \leq \mu$ for suspension type singularities $\{g(x,y)+z^k = 0\} \subset (\mathbb{C}^3,0)$,
- [19] proved $6p_g \leq \mu$ for absolutely isolated singularities.

Moreover, for arbitrary dimension, [29] proved the inequality $\mu \geq (n+1)!\,p_g$ for isolated weighted-homogeneous hypersurface singularities in $(\mathbb{C}^{n+1},0)$. The natural expectation was that the same bound $(n+1)!$ holds for any ICIS of any dimension n and any codimension.

In Section 3 we give a counterexample to the strong conjecture in any codimension $r \geq 2$. In fact, for $r \geq 2$, the bound $(n+1)!$ is wrong even asymptotically. Therefore, in Section 4 we propose a weaker bound. It is based on the Stirling number of the second kind: $\left\{ {n+r \atop r} \right\} := \frac{1}{r!}\sum_{j=0}^{r}(-1)^j\binom{r}{j}(r-j)^{n+r}$, *cf.* [1, Section 24.1.4].

CONJECTURE. *Let $(X,0) \subset (\mathbb{C}^N,0)$ be an ICIS of dimension n and codimension $r = N - n$. Then*

- *for $n = 2$ and $r = 1$ one has $\mu \geq 6p_g$,*
- *for $n = 2$ and arbitrary r one has $\mu > 4p_g$,*
- *for $n \geq 3$ and fixed r one has $\mu \geq C_{n,r} \cdot p_g$, where the coefficient $C_{n,r}$ is defined by*

$$C_{n,r} := \frac{\binom{n+r-1}{n}(n+r)!}{\left\{ {n+r \atop r} \right\} r!}.$$

Note that for any curve singularity (*i.e.*, $n = 1$), p_g is the delta invariant δ and $\mu \leq 2\delta$, (with equality exactly for irreducible ones); see Example 2.1. For $n = 2$ the inequality $\mu(X,0) \geq C_{n,r} \cdot p_g(X,0)$, in general, is not satisfied (see Proposition 4.2 and the comments following it).

We show that the bound of the conjecture is asymptotically sharp, *i.e.*, for any fixed n and r there exists a sequence of isolated complete intersections for which the ratio $\frac{\mu}{p_g}$ tends to $C_{n,r}$. In our subsequent longer article [13], we verify it for several cases; here we exemplify only some 'homogeneous' situations. Moreover, we list some elementary properties of the sequence $\{C_{n,r}\}_{n,r}$, *cf.* Section 4. For example, we show:

$$(1.1)\qquad (n+1)! = C_{n,1} \geq C_{n,2} = \frac{(n+2)!\,(n+1)}{2^{n+2}-2} \geq \cdots \geq \lim_{r\to\infty} C_{n,r} = 2^n.$$

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2. Background.

2.1. The Milnor number. Let $(X,0) \subset (\mathbb{C}^N,0)$ be an isolated complete intersection of dimension $n \geq 1$ defined as the germ of the zero set of an analytic germ $f\colon (\mathbb{C}^N,0) \to (\mathbb{C}^r,0)$. Let $B \subset \mathbb{C}^N$ be a small ball centered at the origin. By the classical theory (see, *e.g.*, [16]) the 'Milnor fiber' $f^{-1}(\epsilon) \cap B$ ($0 < \epsilon \ll$ the radius of B) is smooth and has the homotopy type of a bouquet of n-spheres. Their number μ is called the Milnor number of $(X,0)$.

For general formulae of μ see, *e.g.*, [20] and [16, Section 9.A].

2.2. The singularity (geometric) genus. Let $(X,0) \subset (\mathbb{C}^N,0)$ be an ICIS as above with $n > 1$. Let $(\tilde{X},E) \to (X,0)$ be one of its resolutions, *i.e.*, a birational morphism with $\tilde{X}$ smooth, $\tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{0\}$ and $\overline{\tilde{X} \setminus E} = \tilde{X}$. The singularity genus reflects the cohomological non-triviality of the structure sheaf on the germ $(\tilde{X},E)$, [3]:

$$(2.1)\qquad p_g(X,0) := h^{n-1}\mathcal{O}_{\tilde{X}}.$$

This number does not depend on the choice of the resolution.

EXAMPLE 2.1.

- If $(X,0)$ is a (reduced) curve singularity then the analogue of singularity genus is classical delta invariant: $\delta = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{X}}/\mathcal{O}_{(X,0)}$. It is related to the Milnor number via $2\delta = \mu + \sharp - 1$, where $\sharp$ is the number of local branches; *cf.* [20], [5], [17].
- Assume that $(X,0)$ is a normal surface singularity which admits a smoothing. If F is the Milnor fiber of the smoothing, let (μ_+,μ_0,μ_-) be the Sylvester invariants of the symmetric intersection form in the middle integral homology $H_2(F,\mathbb{Z})$. Then $2p_g = \mu_0 + \mu_+$ [7, Proposition 3.1].
- Let $(X,0)$ be a *Gorenstein* normal surface singularity which admits a smoothing. Then, in fact, the Milnor number of the smoothing is independent of the smoothing. Indeed, if $(\tilde{X},E) \to (X,0)$ is any resolution with relative canonical class $K_{\tilde{X}}$, then $\mu+1 = \chi_{\mathrm{top}}(E) + K^2_{\tilde{X}} + 12p_g$. This formula was proved in [15] for hypersurface surface singularities and in [17] for normal Gorenstein smoothable singularities.

- Let $(X,0) \subset (\mathbb{C}^{n+1},0)$ be an isolated hypersurface singularity. Let $\mathrm{Sp}(X,0) \subset (-1,n)$ be its singularity spectrum. Then p_g is the cardinality of $\mathrm{Sp} \cap (-1,0]$, *cf.* [23], [25].

For the general introduction to singularities we refer to [2], [6], [16], [24].

3. The counterexample. Let $Y = \{f_1 = \cdots = f_r = 0\} \subset \mathbb{P}^{N-1}$ be a smooth projective complete intersection defined by homogeneous polynomials of degrees $\deg(f_i) = p_i$. Define the corresponding ICIS as the cone over Y: $(X,0) = \mathrm{Cone}(Y) \subset (\mathbb{C}^N,0)$.

Proposition 3.1.

$$\mu = \Big(\prod_{i=1}^{r} p_i\Big) \sum_{j=0}^{n} (-1)^j \Big(\sum_{\substack{(k_1,\dots,k_r), k_i \geq 0 \\ \sum_i k_i = n-j}} \prod_{i=1}^{r} (p_i - 1)^{k_i} \Big) - (-1)^n, \tag{1}$$

$$p_g = \sum_{\substack{(k_1,\dots,k_r), k_i \geq 0 \\ \sum_i k_i = n}} \prod_{i=1}^{r} \binom{p_i}{k_i + 1}. \tag{2}$$

For a systematic study of the Milnor number of weighted homogeneous complete intersections, the reader is invited to consult [8], [9], where several formulae were proved. Later, even some more formulae were found, see, *e.g.*, [10], [11]. For formulae regarding the geometric genus we refer to [14], [18], [11]. These formulae usually are rather different than ours considered above. Nevertheless, the above expressions can be derived from them: here, for the Milnor number we will use [9], and for the geometric genus [18].

Proof. (1) We will determine the Euler characteristic $\chi = (-1)^n \mu + 1$ of the Milnor fiber. For a power series $Z := \sum_{i \geq 0} a_i x^i$ write $[Z]_n$ for the coefficient a_n of x^n. Also, set $P := \prod_{i=1}^{r} p_i$. Then, by formula 3.7(c) of [9],

$$\chi = P \cdot \Big[\frac{(1+x)^N}{\prod_i (1+p_i x)} \Big]_n .$$

Rewrite $1 + p_i x$ as $(1+x)(1 - \frac{(1-p_i)x}{1+x})$, hence

$$\begin{aligned}
\Big[\frac{(1+x)^N}{\prod_i (1+p_i x)} \Big]_n &= \Big[(1+x)^n \cdot \prod_i \sum_{k_i \geq 0} \Big(\frac{(1-p_i)x}{1+x} \Big)^{k_i} \Big]_n \\
&= \Big[\sum_{k_1 \geq 0, \dots, k_r \geq 0} x^{\sum k_i} (1+x)^{n - \sum k_i} \prod_i (1-p_i)^{k_i} \Big]_n \\
&= \sum_{\substack{k_1 \geq 0, \dots, k_r \geq 0 \\ \sum k_i \leq n}} \prod_i (1-p_i)^{k_i} .
\end{aligned}$$

(2) By Theorem 2.4 of [18] (and computation of the lattice points under the 'homogeneous Newton diagram') we get

(3.1)
$$p_g = \binom{\sum_k p_k}{N} - \sum_{1\le i\le r}\binom{(\sum_k p_k)-p_i}{N} + \sum_{1\le i<j\le r}\binom{(\sum_k p_k)-p_i-p_j}{N} - \cdots .$$

Using the Taylor expansion $\frac{1}{(1-z)^{N+1}} = \sum_{l\ge N}\binom{l}{N}z^{l-N}$, the right-hand side of (3.1) is
$$\left[\frac{\prod_i(1-z^{p_i})}{(1-z)^{N+1}}\right]_{\sum_k p_k - N}.$$

Thus
$$p_g = \mathrm{rez}_{z=0}\, F(z), \quad \text{where} \quad F(z) := \frac{\prod_i(1-z^{p_i})}{z^{\sum p_i - N+1}(1-z)^{N+1}}.$$

Note that $\mathrm{rez}_{z=\infty} F(z) = 0$ since $F(1/z)/z^2$ is regular at zero. As $F(z)$ is a rational function the sum of its residues vanishes: $\sum_{pt}\mathrm{rez}_{z=pt} F(z) = 0$. Since $F(z)$ has poles at $z=0$ and $z=1$ only, we have $p_g = -\mathrm{rez}_{z=1} F(z)$. By the change of variables $z\mapsto 1/z$ we get
$$p_g = \mathrm{rez}_{z=1}\frac{\prod_{i=1}^r(z^{p_i}-1)}{(z-1)^{N+1}}.$$

Since $z^p - 1 = \sum_{k\ge 0}\binom{p}{k+1}(z-1)^{k+1}$, we obtain
$$p_g = \mathrm{rez}_{z=1}\frac{1}{(z-1)^{n+1}}\cdot\prod_{i=1}^r\sum_{k_i\ge 0}\binom{p_i}{k_i+1}(z-1)^{k_i}$$
$$= \sum_{\substack{k_1,\dots,k_r\ge 0\\ k_1+\cdots+k_r=n}}\binom{p_1}{k_1+1}\cdots\binom{p_r}{k_r+1}. \qquad \square$$

EXAMPLE 3.2. Consider the particular case $p_1 = \cdots = p_r = p$. Then the formulae of Proposition 3.1 read as
$$\mu = (-1)^n\left(p^r\sum_{j=0}^n(1-p)^j\binom{j+r-1}{j} - 1\right), \quad \text{(see also [9, 3.10(b)])},$$

(3.2)
$$p_g = \sum_{\substack{(k_1,\dots,k_r),k_i\ge 0\\ \sum_i k_i=n}}\prod_{i=1}^r\binom{p}{k_i+1}.$$

In some low-dimensional cases we have:
- for $n=1$: $p_g = \frac{r(p-1)p^r}{1!\,2^1}$,
- for $n=2$: $p_g = \frac{r(p-1)p^r}{2!\,2^2}\big(r(p-1)+\frac{p-5}{3}\big)$,

- for $n = 3$: $p_g = \frac{r(p-1)p^r}{3!\,2^3}(pr - 2 - r)(pr - 3 + p - r)$.

Analyzing (3.2) one sees that in the case of μ, the leading term in p (for p large) comes from the last summand ($j = n$) and $\mu = p^N\binom{N-1}{n} + O(p^{N-1})$.

The leading term for p_g is more complicated. For $p \gg 0$ one has:

$$p_g = p^N \cdot \sum_{\substack{(k_1,\ldots,k_r),k_i\geq 0\\ \sum_i k_i = n}} \prod_{i=1}^{r} \frac{1}{(k_i+1)!} + O(p^{N-1}).$$

Thus, asymptotically,

$$\frac{\mu}{p_g} = \frac{\binom{N-1}{n}}{\sum\limits_{\substack{(k_1,\ldots,k_r),k_i\geq 0\\ \sum_i k_i = n}} \prod\limits_{i=1}^{r} \frac{1}{(k_i+1)!}} + O\Big(\frac{1}{p}\Big),$$

which in low dimensions gives:

$$\begin{aligned} n = N - r = 2: &\quad \frac{\mu}{4p_g} = \frac{r+1}{r+\frac{1}{3}} + O\Big(\frac{1}{p}\Big);\\ n = N - r = 3: &\quad \frac{\mu}{8p_g} = \frac{r+2}{r} + O\Big(\frac{1}{p}\Big). \end{aligned} \tag{3.3}$$

Since $4(r+1)/(r+\frac{1}{3}) < 6$ for all $r \geq 2$, in the case $n = 2$ the strong Durfee bound is violated even asymptotically—that is, for any p sufficiently large—whenever $r \geq 2$. Similarly, for $n = 3$, $8(r+2)/r < 4!$ whenever $r \geq 2$.

4. **The new conjecture with the new weaker bound.** In the previous section we have seen that for the singularity which is the cone over a smooth projective complete intersection with $p_1 = \cdots = p_r$, the ratio μ/p_g tends to the numerical factor

$$C_{n,r} := \frac{\binom{N-1}{n}}{\sum\limits_{\substack{(k_1,\ldots,k_r),k_i\geq 0\\ \sum_i k_i = n}} \prod\limits_{i=1}^{r} \frac{1}{(k_i+1)!}}. \tag{4.1}$$

Next we list some properties of these numbers.

LEMMA 4.1.
(1)

$$C_{n,r} = \frac{\binom{n+r-1}{n}(n+r)!}{\left\{ {n+r \atop r!} \right\}}$$

(see also [12, pp. 176–178]*).*

(2) $(n+1)! = C_{n,1} \geq C_{n,2} \geq \cdots \geq \lim_{r\to\infty} C_{n,r} = 2^n$.

PROOF. (1) Let $S_{n,r} := \sum_{\substack{\{k_i\geq 0\}\\ \sum_i k_i = n}} \prod_{i=1}^{r} \frac{1}{(k_i+1)!}$. First we prove

$$\frac{r^{n+r}}{(n+r)!} = S_{n,r} + rS_{n+1,r-1} + \binom{r}{2} S_{n+2,r-2} + \cdots . \tag{4.2}$$

Consider the expansion

$$r^{n+r} = (1+\cdots+1)^{n+r} = \sum_{k_1,\ldots,k_r\geq 0} \Big(\prod_{i=1}^{r} 1^{k_i}\Big) \frac{(n+r)!}{\prod_{i=1}^{r} k_i!}.$$

This gives

$$\frac{r^{n+r}}{(n+r)!} = \sum_{\substack{k_1,\ldots,k_r\geq 0\\ \sum_i k_i = n+r}} \frac{1}{\prod_{i=1}^{r} k_i!}. \tag{4.3}$$

For each $0 \leq j \leq r$ and subset $I(j) \in \{1,\ldots,r\}$ of cardinality j set

$$K_{I(j)} := \Big\{ \underline{k} = (k_1,\ldots,k_r) \text{ with } k_i = 0 \text{ for } i \in I(j) \text{ and } \sum_i k_i = n+r \Big\}.$$

Then the sum $\sum_{\underline{k}}$ from the right-hand side of (4.3) can be replaced by

$$\sum_{j=0}^{r} \sum_{I(j)} \sum_{\underline{k}\in K_{I(j)}} \frac{1}{\prod_{i=1}^{r} k_i!}.$$

Since for each fixed $I(j)$

$$\sum_{\underline{k}\in K_{I(j)}} \frac{1}{\prod_{i=1}^{r} k_i!} = S_{n+j,r-j}$$

(by a change $k_j \mapsto k_j - 1$ for those indices when $k_j \neq 0$), and for each j there are $\binom{r}{j}$ subsets $I(j)$ of cardinality j, (4.2) follows.

Now, we prove that $S_{n,r} = \frac{\sum_{i=0}^{r}(r-i)^{n+r}(-1)^i\binom{r}{i}}{(n+r)!}$. Note that the recursion (4.2) defines $S_{n,r}$ uniquely. Hence it is enough to verify that the proposed solution satisfies this recursive relation; that is, $r^{n+r} = \sum_{j=0}^{r}\binom{r}{r-j}\sum_{i=0}^{j}(j-i)^{n+r}(-1)^i\binom{j}{i}$. Set $\tilde{j} = j - i$. Then the sum can be rewritten as $\sum_{\tilde{j}=0}^{r} \tilde{j}^{n+r}\binom{r}{\tilde{j}}\sum_{i=0}^{r-\tilde{j}}(-1)^i\binom{r-\tilde{j}}{i}$. Note that $\sum_{i=0}^{r-\tilde{j}}(-1)^i\binom{r-\tilde{j}}{i} = (1-1)^{r-\tilde{j}} = 0$ unless $r = \tilde{j}$. Hence the above formula for $S_{n,r}$ follows.

This formula can be compared with that one satisfied by the Stirling numbers (see introduction), hence we get $S_{n,r} = \left\{ {n+r \atop r} \right\} \frac{r!}{(n+r)!}$, which proves the statement for $C_{n,r}$.

(2) The limit of $C_{n,r}$ is computed using the asymptotics of Stirling numbers of the second kind, [1, Section 24.1.4]: $\left\{ {n+r \atop r} \right\} \sim \frac{r^{2n}}{2^n n!}$. This gives $C_{n,r} \sim 2^n \frac{(n+r-1)!\,(n+r)!}{(r-1)!\,r!\,r^{2n}} \to 2^n$.

The following proof, showing that $C_{n,r}$ is non-increasing sequence in r, was communicated to us by D. Moews. Write the inequality $C_{n,r} \geq C_{n,r+1}$ in terms of Stirling numbers of the second kind:

$$\left\{ {n+r+1 \atop r+1} \right\} r(r+1) \geq \left\{ {n+r \atop r} \right\} (n+r)(n+r+1). \tag{4.4}$$

Set $N := n+r$ as usual, and write the inequality in terms of generating functions:

$$\frac{r(r+1)}{x} \sum_{N \geq r} \left\{ {N+1 \atop r+1} \right\} \frac{x^{N+1}}{(N+1)!} \succeq x\partial_x \sum_{N \geq r} \left\{ {N \atop r} \right\} \frac{x^N}{N!} \tag{4.5}$$

with the convention $\sum a_n x^n \succeq \sum b_n x^n$, if and only if $a_n \geq b_n$ for each $n \geq 0$.

Recall that Stirling numbers are the coefficients of Taylor series: $\frac{(e^x-1)^r}{r!} = \sum_{N \geq r} \left\{ {N \atop r} \right\} \frac{x^N}{N!}$. Therefore the inequality to be proved can be rewritten as

$$\frac{(e^x-1)^{r+1}}{(r-1)!\,x} \succeq \frac{xe^x(e^x-1)^{r-1}}{(r-1)!}.$$

We claim that this inequality follows from $(e^x-1)^2 \succeq x^2 e^x$. Indeed, if $\sum a_n x^n \succeq \sum b_n x^n$ and the Taylor expansion of $g(x)$ has only positive coefficients, then $g(x) \sum a_n x^n \succeq g(x) \sum b_n x^n$. Hence it is enough to prove $(e^x-1)^2 \succeq x^2 e^x$. For the same reason, this last statement will follow from $(e^x-1) \succeq xe^{\frac{x}{2}}$, which is immediate. □

The coefficients $C_{n,r}$ satisfies the following identities and inequalities (some more will be listed in [13], where the general inequality $\mu \geq C_{n,r} p_g$ will also be discussed).

Proposition 4.2. *Let $(X,0) = \bigcap_{i=1}^r \{f_i(x_1,\dots,x_N) = 0\} \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity, where each f_i is homogeneous of degrees p_i. Set $P := \prod_{i=1}^r p_i$. Then:*

- *for $n=1$, $\mu + P - 1 = C_{1,r} \cdot p_g = 2p_g$ (here, $p_g := \delta$);*
- *for $n=2$, $\mu + P \cdot \left(\frac{r-1}{3r+1} \sum_{i=1}^r (p_i - 1) - \sum_{i<j} \frac{(p_i-p_j)^2}{3r+1} - 1 \right) + 1 = C_{2,r} \cdot p_g = 4\frac{r+1}{r+\frac{1}{3}} \cdot p_g$;*
- *for $r=1$, $\mu - C_{n,1} \cdot p_g = (p-1)^N - \frac{p!}{(p-N)!} \geq 0$. In particular, $\mu \geq (n+1)! \cdot p_g$;*
- *for $n=3$, $\mu > C_{3,r} \cdot p_g$.*

We wish to emphasize the equation $\mu + P\cdot E + 1 = C_{2,r}\cdot p_g$ in the case $n=2$. Note that the expression E is -1 (hence negative) if $r=1$, but for $r\geq 2$ and some choices of p_i's (*e.g.*, whenever they are all equal) E might be positive.

PROOF. By Proposition 3.1, we have

$$\begin{aligned}\mu &= P\cdot\sum_{j=0}^{n}(-1)^j\Big(\sum_{\substack{\{k_i\geq 0\},\\ \sum_i k_i=n-j}}\prod_{i=1}^{r}(p_i-1)^{k_i}\Big)-(-1)^n,\\ p_g &= P\cdot\sum_{\substack{\{k_i\geq 0\}\\ \sum_i k_i=n}}\prod_{i=1}^{r}\binom{p_i-1}{k_i}\frac{1}{k_i+1}.\end{aligned} \tag{4.6}$$

We can assume $p_i>1$, otherwise some of the defining hypersurfaces are hyperplanes, so this reduces to a singularity of *the same dimension* and smaller codimension.

CASE $n=1$. Use

$$\mu = P\cdot\Big(\sum_{i=1}^{r}(p_i-1)-1\Big)+1,\quad p_g=P\cdot\sum_{i=1}^{r}\frac{(p_i-1)}{2},\quad C_{1,r}=2. \tag{4.7}$$

CASE $n=2$. We have

$$\begin{aligned}\mu &= P\cdot\Big(\sum_i(p_i-1)(p_i-2)+\sum_{i<j}(p_i-1(p_j-1)+1\Big)-1,\\ p_g &= P\cdot\Big(\sum_i\binom{p_i-1}{2}\frac{1}{3}+\sum_{i<j}\frac{(p_i-1)(p_j-1)}{4}\Big),\quad C_{2,r}=4\frac{r+1}{r+\frac{1}{3}}.\end{aligned} \tag{4.8}$$

Hence

$$\begin{aligned}\mu - C_{2,r}p_g &= P\cdot\Big(\sum_i(p_i-1)(p_i-2)\frac{(r-1)}{3r+1}\\ &\qquad -\frac{2}{3r+1}\sum_{i<j}(p_i-1)(p_j-1)+1\Big)-1\\ &= P\cdot\Big(\sum_{i<j}\frac{(p_i-1)^2-2(p_i-1)(p_j-1)+(p_j-1)^2}{3r+1}\\ &\qquad -\sum_i(p_i-1)\frac{(r-1)}{3r+1}+1\Big)-1,\end{aligned} \tag{4.9}$$

giving the needed equality. Note that from here one also obtains for $r = 1$ and $n = 2$:

$$6p_g = \mu - P + 1 \leq \mu. \tag{4.10}$$

CASE $r = 1$. This is obvious.

CASE $n = 3$. It is enough to prove

$$\sum_{j=0}^{3}(-1)^j\Big(\sum_{\substack{\{k_i \geq 0\},\\ \sum_i k_i = 3-j}}\prod_{i=1}^{r}(p_i-1)^{k_i}\Big) > \frac{8(r+2)}{r}\sum_{\substack{\{k_i \geq 0\}\\ \sum_i k_i = 3}}\prod_{i=1}^{r}\binom{p_i-1}{k_i}\frac{1}{k_i+1}. \tag{4.11}$$

Present the difference as follows:

$$\begin{aligned} &\underbrace{\sum_{\substack{\{k_i \geq 0\},\\ \sum_i k_i = 3}}\Big(\prod_{i=1}^{r}(p_i-1)^{k_i}\Big)\Big(1-\frac{8(r+2)}{r\prod_{i=1}^{r}(k_i+1)!}\Big)}_{\text{Part I}} \\ &\quad + \underbrace{\frac{8(r+2)}{r}\sum_{\substack{\{k_i \geq 0\}\\ \sum_i k_i = 3}}\Big(\prod_{i=1}^{r}\frac{(p_i-1)^{k_i}}{(k_i+1)!} - \prod_{i=1}^{r}\binom{p_i-1}{k_i}\frac{1}{k_i+1}\Big)}_{\text{Part II}} \\ &\quad + \underbrace{\sum_{j=1}^{3}(-1)^j\Big(\sum_{\substack{\{k_i \geq 0\},\\ \sum_i k_i = 3-j}}\prod_{i=1}^{r}(p_i-1)^{k_i}\Big)}_{\text{Part III}}. \end{aligned} \tag{4.12}$$

First we prove that Part I ≥ 0.

$$\begin{aligned} \text{Part I} = \frac{2r-2}{3r}\sum_i(p_i-1)^3 + \frac{r-4}{3r}\sum_{i \neq j}(p_i-1)^2(p_j-1) \\ - \frac{2}{r}\sum_{i<j<k}(p_i-1)(p_j-1)(p_k-1). \end{aligned} \tag{4.13}$$

Now use the elementary inequalities (the algebraic mean compared to the geometric mean)

$$\sum a_i^n = \sum_{i \neq j}\frac{la_i^n + (n-l)a_j^n}{n(r-1)} \geq \frac{1}{r-1}\sum_{i \neq j}a_i^l a_j^{n-l}$$

and (for $2k < n$)

$$\sum_{i\neq j} a_i^k a_j^{n-k} = \sum_{i\neq j\neq m} \frac{(n-k-1)a_i^k a_j^{n-k} + a_i^k a_m^{n-k}}{(n-k)(r-2)} \geq \frac{\sum_{i\neq j\neq m} a_i^k a_j^{n-k-1} a_m}{r-2}$$

to get

$$\text{(4.14)}\quad \text{Part I} \geq \frac{2+r-4}{3r}\sum_{i\neq j}(p_i-1)^2(p_j-1) - \frac{2}{r}\sum_{i<j<k}(p_i-1)(p_j-1)(p_k-1) \geq 0.$$

Now simplify Part II:

$$\text{(4.15)}\qquad \text{Part II} = \frac{8(r+2)}{r}\Big(\sum_i \frac{(p_i-1)(3p_i-5)}{4!} + \sum_{i\neq j}\frac{(p_i-1)}{3!}\frac{(p_j-1)}{2}\Big).$$

Thus, by direct check one gets Part II + Part III > 0. Hence the statement. □

Finally, we add a slightly weaker inequality, but which is valid for any n (including $n = 2$) and any r. Since $\binom{N-1}{n}$ is the number of terms in the sum from the denominator of $C_{n,r}$, we obtain that

$$C_{n,r} \geq \min_{\substack{\sum k_i = n\\ k_i\geq 0}} \Big\{\prod_{i=1}^r (k_i+1)!\Big\}.$$

PROPOSITION 4.3. *Let $(X,0)$ as in Lemma 4.2 and $n \geq 2$. Then*

$$\mu \geq \min_{\substack{\sum k_i = n\\ k_i\geq 0}} \Big\{\prod_{i=1}^r (k_i+1)!\Big\} \cdot p_g \geq 2^n \cdot p_g.$$

Moreover, if $n > r$, let $n = n_1 r + r_1$ for $n_1, r_1 \geq 0$, $r_1 < r$. Then $\mu > \big((n_1+1)!\big)^{r-r_1}\big((n_1+2)!\big)^{r_1} p_g$.

In particular, for $r = 1$, $\mu > (n+1)!\, p_g$. For $r = 2$ and n-odd, $\mu \geq (\lfloor \frac{n}{2}\rfloor + 1)!\,(\lfloor \frac{n}{2}\rfloor + 2)!\, p_g$. For $r = 2$ and n-even: $\mu \geq \big((\lfloor \frac{n}{2}\rfloor + 1)!\big)^2 p_g$.

PROOF.

STEP 1. We claim that (by Proposition 3.1) it is enough to prove the inequality

$$\text{(4.16)}\qquad \text{LHS} := P\cdot\Big(\sum_{j=0}^n (-1)^j \sum_{\substack{\{k_i\geq 0\}\\ \sum_i k_i = n-j}} \prod_{i=1}^r (p_i-1)^{k_i}\Big) - (-1)^n$$

$$> \sum_{\substack{k_1,\dots,k_r\geq 0\\ k_1+\dots+k_r = n}} \Big(\prod_{i=1}^r \binom{p_i}{k_i+1}(k_i+1)!\Big) =: \text{RHS}.$$

Clearly this implies the first inequality of the proposition, while the second follows from $(k+1)! \geq 2^k$.

STEP 2. The left-hand side can be written as

(4.17)

$$\text{LHS} = (p_r - 1)\Big(\prod_{i=1}^{r-1} p_i\Big) C_n(p_1, \dots, p_r) + (p_{r-1} - 1)\Big(\prod_{i=1}^{r-2} p_i\Big) C_n(p_1, \dots, p_{r-1}) + \dots + (p_1 - 1)(p_1 - 1)^n,$$

where

$$C_n(p_1, \dots, p_r) := \sum_{\substack{\{k_i \geq 0\} \\ \sum_i k_i = n}} \Big(\prod_{i=1}^{r} (p_i - 1)^{k_i}\Big).$$

This is expansion in terms with decreasing r, hence it is natural to expand the right-hand side similarly.

Let $D_n(p_1, \dots, p_r) := \sum_{\substack{k_1, \dots, k_r \geq 0 \\ k_1 + \dots + k_r = n}} \prod_{i=1}^{r} \binom{p_i - 1}{k_i} (k_i)!$. So $D_1(p_1) = \binom{p_1 - 1}{n} n!$. Write the RHS as follows:

$$\Big(\Big(\prod_{i=1}^{r} p_i\Big) D_n(p_1, \dots, p_r) - \Big(\prod_{i=1}^{r-1} p_i\Big) D_n(p_1, \dots, p_{r-1})\Big) + \Big(\Big(\prod_{i=1}^{r-1} p_i\Big) D_n(p_1, \dots, p_{r-1}) - \Big(\prod_{i=1}^{r-2} p_i\Big) D_n(p_1, \dots, p_{r-2})\Big) + \dots + p_1 D_1(p_1). \tag{4.18}$$

Thus it is enough to prove the inequality for each pair of terms in these expansions:

(4.19)

$$\forall r \geq 1,\ n \geq 2: \quad (p_r - 1)\Big(\prod_{i=1}^{r-1} p_i\Big) C_n(p_1, \dots, p_r) \geq \Big(\prod_{i=1}^{r} p_i\Big) D_n(p_1, \dots, p_r) - \Big(\prod_{i=1}^{r-1} p_i\Big) D_n(p_1, \dots, p_{r-1}).$$

For example, for $r = 1$ we have $(p_1 - 1)^{n+1} > p_1 \binom{p_1 - 1}{n} n!$. Present the right-hand side of (4.19) in the form

$$(p_r - 1)\Big(\prod_{i=1}^{r-1} p_i\Big) D_n(p_1, \dots, p_r) + \Big(\prod_{i=1}^{r-1} p_i\Big)\big(D_n(p_1, \dots, p_r) - D_n(p_1, \dots, p_{r-1})\big).$$

Note that

(4.20)
$$D_n(p_1,\dots,p_r)-D_n(p_1,\dots,p_{r-1})=\sum_{\substack{\{k_i\geq 0\},k_r>0\\ k_1+\dots+k_r=n}}\prod_{i=1}^{r}\binom{p_i-1}{k_i}(k_i)!$$
$$=(p_r-1)D_{n-1}\big(p_1,\dots,p_{r-1},(p_r-1)\big).$$

Hence we have to prove the inequality

$$C_n(p_1,\dots,p_r)\geq D_n(p_1\dots,p_r)+D_{n-1}\big(p_1,\dots,p_{r-1},(p_r-1)\big).$$

In fact, we will prove by induction on r the stronger inequality:

$$C_n(p_1,\dots,p_r)\geq D_n(p_1,\dots,p_r)+D_{n-1}(p_1,\dots,p_r). \tag{4.21}$$

STEP 3. Both parts are summations over $(k_1,\dots,k_r)$; expand them in k_r. Then we have to prove:

(4.22)
$$\sum_{k_r=0}^{n}\left((p_r-1)^{k_r}C_{n-k_r}(p_1,\dots,p_{r-1})-\binom{p_r-1}{k_r}k_r!\,D_{n-k_r}(p_1,\dots,p_{r-1})\right)$$
$$\geq\sum_{k_r=1}^{n}\binom{p_r-1}{k_r-1}(k_r-1)!\,D_{n-k_r}(p_1,\dots,p_{r-1}).$$

For $k_r\geq 2$, one has $(p_r-1)^{k_r}\geq\binom{p_r-1}{k_r}k_r!+\binom{p_r-1}{k_r-1}(k_r-1)!$, by direct check. And $C_j(p_1,\dots,p_{r-1})\geq D_j(p_1,\dots,p_{r-1})$, for all values of j, r. Therefore we only need to check the terms for $k_r=0,1$, *i.e.*, it is enough to prove:

(4.23)
$$(p_r-1)\big(C_{n-1}(p_1,\dots,p_{r-1})-D_{n-1}(p_1,\dots,p_{r-1})\big)$$
$$+C_n(p_1,\dots,p_{r-1})-D_n(p_1,\dots,p_{r-1})\geq D_{n-1}(p_1,\dots,p_{r-1}).$$

Again, $C_{n-1}(p_1,\dots,p_{r-1})\geq D_{n-1}(p_1,\dots,p_{r-1})$, so the initial inequality reduces to $C_n(p_1,\dots,p_{r-1})\geq D_n(p_1,\dots,p_{r-1})+D_{n-1}(p_1,\dots,p_{r-1})$. This completes the induction step from $(r-1)$ to r.

Finally, for $r=1$ the initial inequality is $(p_1-1)^n\geq\binom{p_1-1}{n}n!+\binom{p_1-1}{n-1}(n-1)!$. By direct check it is true for $n\geq 2$. □

NOTE IN PRINT: In our subsequent paper arxiv:1111.1411, we prove the conjecture (formulated on page 2) for homogeneous complete intersections.

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