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Compactifications of $\mathbb{C}^2$ via Pencils of Jets of Curves

Pinaki Mondal

Presented by Pierre Milman, FRSC

Abstract. This article is mainly an announcement of some of the results in the articles *Primitive normal compactifications of the affine plane I* and *II* where we study normal compactifications of the affine plane from the point of view of associated pencils of jets of curves and corresponding valuations on the field of rational functions. We find an explicit criterion to determine if a discrete valuation corresponds to a normal compactification of $\mathbb{C}^2$ which is *primitive* (*i.e.*, the curve at infinity is irreducible). We show that a primitive normal compactification of $\mathbb{C}^2$ is projective iff it is algebraic iff the associated pencil of jets of curves has a representative which has only one place at infinity. As an application we compute the moduli space of primitive projective compactifications of $\mathbb{C}^2$. We also characterize primitive normal compactifications of $\mathbb{C}^2$ which are not algebraic.

Résumé. Cet article est principalement une annonce de certains résultats dans les articles *Primitive normal compactifications of the affine plane I* et *II* où l'on étudie les compactifications normales du plan affine du point de vue des pinceaux associés avec les jets de courbes et des valuations correspondant sur le corps des fonctions rationnelles. Nous trouvons un critère explicite pour déterminer si une valuation discrète correspond à une compactification normale de $\mathbb{C}^2$ qui est *primitive* (*i.e.*, la courbe à l'infini est irréductible). Nous montrons qu'une compactification normale primitive de $\mathbb{C}^2$ est projective si et seulement si elle est algébrique si et seulement si le pinceau associé de jets de courbes a un représentant qui n'a qu'un seul endroit à l'infini. Comme application nous calculons l'espace des modules des compactifications primitives projectives de $\mathbb{C}^2$. Nous avons également caractérisé les compactifications primitives normales de $\mathbb{C}^2$ qui ne sont pas algébriques.

1. **Introduction.** Every curve at infinity on a normal compactification of $\mathbb{C}^2$ (*i.e.*, a normal compact analytic surface containing $\mathbb{C}^2$) induces a discrete valuation on $\mathbb{C}(x, y)$, namely the *order of vanishing* along the curve. It is well known that discrete valuations on $\mathbb{C}(x, y)$ correspond to Puiseux series (see *e.g.*, [FJ04, Chapter 4] for a modern treatment). Our starting point in [Mon11a] is a reformulation of the Puiseux series treatment in terms of pencils of *jets of curve-germs* (see Definition 2.2), whose generic elements intersect the corresponding

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curves at infinity at distinct points. The geometries of a normal compactification X of $\mathbb{C}^2$ and its associated pencils of curve-germs determine each other (*e.g.*, the resolution graph of X corresponds in a natural way to the resolution graph of the *curvettes* in the pencils) and it is a goal of this project to extract information about the former in terms of the latter.

Note . From now on, for the sake of avoiding repetitions, we will omit the word 'normal' when talking about compactifications of $\mathbb{C}^2$: all the compactifications will be normal unless otherwise stated.

Compactifications of $\mathbb{C}^2$ have been studied by a number of authors from different perspectives. *E.g.*, in [Mor72] and [Bre73] the interest was in classifying the curve at infinity on *non-singular* compactifications of $\mathbb{C}^2$; in [Bre80], [BDP81], [MZ88] classes of compactifications of $\mathbb{C}^2$ were studied as examples of surfaces with prescribed singularities; in [Fur97], [Oht01] the subject was to classify *primitive compactifications* of $\mathbb{C}^2$ (*i.e.*, those for which the curve at infinity is irreducible) which can be embedded into $\mathbb{P}^3$; the topic of [Koj01] and [KT09] was primitive compactifications of $\mathbb{C}^2$ with prescribed singularities; more recently compactifications of $\mathbb{C}^2$ were applied in [FJ11] to study *polynomial dynamics* of $\mathbb{C}^2$. Our interest in this subject stems from the study of affine Bezout type theorems, since an understanding of the structure of compactifications of an affine variety can pave the way to find number of solutions of polynomial systems on it (*e.g.*, toric compactifications of $(\mathbb{C}^*)^n$ yield theorems of Kushirenko [Kus76] and Bernstein [Ber75]).

In [Mon11a] and [Mon11b] we mainly consider primitive compactifications of $\mathbb{C}^2$, because they are in some sense the simplest, and because primitive *projective* compactifications correspond to the simplest non-trivial classes of *iterated semidegrees* which were studied in [Mon10] in relation to affine Bezout type theorems. We give an explicit numerical characterization of the pencils of germs of curve-jets which correspond to primitive compactifications of $\mathbb{C}^2$ (Theorem 3.6) and consequently a characterization of the resolution graphs of these surfaces (Corollary 3.7). In [Mon11a] we also calculate some invariants of these surfaces, *e.g.*, we calculate the canonical class and determine the precise type of singularity corresponding to the *zeroth* jet of the pencil.

Let X be a primitive compactification of $\mathbb{C}^2$. The *local* geometry of X near the curve at infinity is determined by the germs in the associated pencil, but the *global* geometry of X depends on global representatives of the pencil. In particular, we show that a primitive normal compactification of $\mathbb{C}^2$ is projective iff it is algebraic iff one of the curvettes in the associated pencil can be represented by a plane curve with only one place at infinity (Theorem 3.8). Both primitive projective compactifications of $\mathbb{C}^2$ and plane curves with one place at infinity are much studied subjects (see *e.g.*, [AM73], [AM75], [Gan79], [Rus80], [NO97], [Suz99] for the latter), and we believe the connection we found between these two objects will be useful for the study of both. In particular it would be interesting to relate our results to that of [CPR05] (who also consider a class of compactifications associated to a particular type of pencils of curvettes) and [FJ07].

As an application of Theorem 3.8 we compute explicit equations of primitive projective compactifications of $\mathbb{C}^2$ from the Puiseux series of the associated pencil of curve-germs (Corollary 3.11) and calculate the moduli space of primitive projective compactifications of $\mathbb{C}^2$ with a fixed *normal form* of Puiseux pairs (Corollary 3.14). The defining equations and moduli space of primitive projective compactifications correspond in a natural way to the equations and moduli space of plane curves with one place at infinity, the latter being studied in [Oka98] and [FS02]. In particular, it follows from the defining equations that the curve at infinity is non-singular. Moreover, the preceding results give a characterization of primitive compactifications of $\mathbb{C}^2$ which are *not* algebraic.[1]

Fix polynomial coordinates (x, y) on $\mathbb{C}^2$. Then for every primitive compactification X of $\mathbb{C}^2$, the order of vanishing along the curve at infinity induces a discrete valuation ν on $\mathbb{C}(x, y)$. It is straightforward to see that

$$(*) \qquad \nu(f) < 0 \quad \text{for all } f \in \mathbb{C}[x, y] \setminus \mathbb{C}, \text{ and } \nu \text{ is divisorial,}$$

i.e., the transcendence degree of the valuation ring of ν over $\mathbb{C}$ is 1. Moreover, it is not hard to see (*e.g.*, by the universal property of normal varieties) that ν uniquely determines X, *i.e.*, if Z is another primitive normal compactification of $\mathbb{C}^2$ such that the order of vanishing along $Z \setminus \mathbb{C}^2$ equals ν, then $Z \cong X$. This gives rise to a natural question:

QUESTION 1.1. Does every discrete valuation on $\mathbb{C}(x, y)$ that satisfies $(*)$ come from a primitive compactification of $\mathbb{C}^2$?

One of the motivating factors of this study was to understand Question 1.1 and we give a complete answer (see Remark 3.18).

Our article is organized as follows. In the next section we set up the preliminary notions of jets of curve-germs and reformulate the Puiseux series treatment of discrete valuations on $\mathbb{C}(x, y)$ in terms of curvettes. We apply this reformulation to find an upper bound on number of singular points on compactification of $\mathbb{C}^2$ and a description of resolution graphs of primitive compactifications of $\mathbb{C}^2$. In Section 3 we state our main results. Section 4 contains our conjecture on algebraic contraction of curves. We also give an explicit example of two primitive compactifications of $\mathbb{C}^2$ with one singular point such that they have the same resolution graph, yet one is algebraic and the other is not.

I heartily thank Professor Pierre Milman. This work is essentially an attempt to understand some of his questions in a simple case and the exposition profited enormously from speaking in his weekly seminar and from his questions. Very special thanks also go to Dmitry Kerner—his questions *forced* me to think

[1]The non-algebraic normal completions of $\mathbb{C}^2$ provide examples of non-algebraic normal Moishezon surfaces that can be constructed from from blowing down *rational* curves from rational surfaces. This phenomenon does not seem to be recorded in the literature, *e.g.*, the constructions in [Nag58] and [Gra62] of non-algebraic normal Moishezon surfaces involve blowing down *non-rational* curves.

and formulate the results in geometric and much more understandable terms; examples include formulations of compactifications in terms of curvettes (I also learned of the word 'curvette' from him) and the geometric interpretation of Lemma 3.19.

2. Valuations and pencils of curvettes. Our approach to understand valuations was based on Puiseux series associated with them, and this led us to consider associated pencils of curvettes (to be defined below) which, although formally equivalent to the Puiseux series description, are perhaps more *geometric* (*cf.* Proposition 2.8 and the proof of Proposition 2.9).

Definition 2.1 (Meromorphic Puiseux series). A *Meromorphic Puiseux series* in a variable u is a fractional power series of the form $\sum_{l \geq k} a_l u^{l/m}$ for some $k, m \in \mathbb{Z}$, $m \geq 1$ and $a_l \in \mathbb{C}$ for all $l \in \mathbb{Z}$. If all exponents of u appearing in a meromorphic Puiseux series are positive, then it is simply called a *Puiseux series* (in u). Given a meromorphic Puiseux series $\phi(u)$ in u, write it in the following form:

$$\phi(u) = \cdots + a_1 u^{\frac{q_1}{p_1}} + \cdots + a_2 u^{\frac{q_2}{p_1 p_2}} + \cdots + a_l u^{\frac{q_l}{p_1 p_2 \cdots p_l}} + \cdots$$

where q_1/p_1 is the first non-integer exponent, and for each k, $1 \leq k \leq l$, we have that $p_k \geq 2$, $\gcd(p_k, q_k) = 1$, and the exponents of all terms with order less than $\frac{q_k}{p_1 \cdots p_k}$ belong to $\frac{1}{p_1 \cdots p_{k-1}}\mathbb{Z}$. Then the pairs $(q_1, p_1), \ldots, (q_l, p_l)$, are called the *Puiseux pairs* of ϕ and the exponents $q_k/p_1 \cdots p_k$, $1 \leq k \leq l$ are called *characteristic exponents* of ϕ. The *polydromy order* of ϕ is $p_1 \cdots p_l$.

Definition 2.2. Let O be the origin in $\mathbb{C}^2$ and (u, v) be analytic coordinates near O. Given a positive rational number ω and two irreducible analytic germs C_1, C_2 of curves at O, we say that $C_1 \equiv_{u,\omega} C_2$ iff there are Puiseux series expansions $v = \phi_i(u)$ for C_i, $1 \leq i \leq 2$, such that $\operatorname{ord}_u(\phi_1 - \phi_2) > \omega$. A *$(u, \omega)$-jet* $\mathcal{J}$ of curve germs at O is an equivalence class of analytic germs at O modulo the equivalence relation $\equiv_{u,\omega}$. A curve-germ C representing $\mathcal{J}$ is called a *curvette*, if it also satisfies the following condition:

(∗∗) All the characteristic exponents of the Puiseux expansion of C in u are less than or equal to ω.

Note that (∗∗) is simply the algebraic way of saying that C is the 'least singular' of all representatives of $\mathcal{J}$. Frequently we will simply say ω-jet or ω-curvette when the variable u for the Puiseux series expansions is clear from the context. By a *pencil of ω-jets* at O we mean a family $\mathcal{L}$ of curve-germs of the form $\{[\phi(u) + \xi u^{\omega}] : \xi \in \mathbb{C}\}$ for a *finite* Puiseux series ϕ in u such that $\operatorname{ord}_u(\phi) > 0$ and $\deg_u(\phi) < \omega$ (note that here we identified curves with their Puiseux series expansions in u). If the polydromy order of ϕ is p, then let p' be the smallest positive integer such that $q' := pp'\omega \in \mathbb{Z}$. Then the *formal Puiseux pairs* of $\mathcal{L}$ are the Puiseux pairs of ϕ with the pair (q', p') appended at the end. We say that (q', p') is the *generic* formal Puiseux pair for $\mathcal{L}$.

REMARK 2.3. Our definition of curvettes is equivalent to those in [MF11] and [CPR05].

A simple reformulation of the theory of discrete valuations and Puiseux series (as developed in [FJ04]) yields the following correspondences (recall that a valuation ν is *centered* at a point P iff the maximal ideal of the valuation ring of ν contains the maximal ideal of P):

THEOREM 2.4 (*cf.* [FJ04, Proposition 4.1, Theorem 4.17]). *There is a one-to-one correspondence among the following three families:*

(i) *Divisorial discrete valuations ν on $\mathbb{C}(u,v)$ centered at the origin and* normalized, *in the sense that* $\gcd\big(\nu(f) : f \in \mathbb{C}(u,v)\big) = 1$.
(ii) *Finite Puiseux series ϕ in u up to conjugacy and a rational number r such that* $\operatorname{ord}_u(\phi) > 0$ *and* $r > \deg_u(\phi)$.
(iii) *Pencils $\mathcal{L}$ of r-jets of curve-germs at the origin for $r \in \mathbb{Q}$, $r > 0$.*

The correspondence (ii) *to* (i) *is given by $(\phi, r) \mapsto \nu_{\phi,r}$, where for every $f \in \mathbb{C}(u,v)$,*

$$\nu_{\phi,r}(f) := p \operatorname{ord}_u\Big(f\big(u, \phi(u) + \xi u^r\big)\Big),$$

where ξ is an indeterminate *and p is the lowest common multiple of denominators of r and exponents of u in ϕ. The correspondence* (iii) *to* (i) *is given by $\mathcal{L} \mapsto \nu_{\mathcal{L}}$, where for every $f \in \mathbb{C}[u,v]$,*

$$\nu_{\mathcal{L}}(f) := \textit{intersection multiplicity at the origin of } V(f)$$
$$\textit{and a generic } \text{curvette} \textit{ in } \mathcal{L}.$$

EXAMPLE 2.5. If ν is the *order of vanishing* at O and (u,v) are analytic coordinates at O, then the pencil corresponding to ν is $\{\xi u : \xi \in \mathbb{C}\}$. Let $\mathcal{L}' := \{au + \xi u^2 : \xi \in \mathbb{C}\}$ for some $a \in \mathbb{C}^*$. Then $\mathcal{L}'$ corresponds to the order along a point on the exceptional divisor of the blow up of the origin.

Let U be a neighborhood (either in analytic or algebraic sense) of O. We say that $\tilde{U}$ is a *birational lift* of U if there is a regular birational map $\pi \colon \tilde{U} \to U$. We say that a curve C on $\tilde{U}$ *realizes* a divisorial valuation ν iff ν is precisely the order of vanishing along C. Since the order of vanishing function uniquely determines a curve on a surface, and since $\mathbb{C}^2$ is normal, the following lemma is almost immediate:

LEMMA 2.6. *Let ν be a divisorial discrete valuation centered at O. In any birational lift of a neighborhood of O, there is at most one curve that realizes ν. If $\tilde{U}_1$ and $\tilde{U}_2$ are birational lifts of a neighborhood of O and C_i on $\tilde{U}_i$ realizes ν, then for generic $z \in C_1$, the birational map $\tilde{U}_1 \dashrightarrow \tilde{U}_2$ maps z to a point on C_2 and restricts to an isomorphism near z.* ☐

We say that a curve C on a birational lift $\tilde{U}$ of U *separates* a pencil $\mathcal{L}$ of ω-jets at O, iff C maps to O and strict transforms in $\tilde{U}$ of (representatives of) curves representing generic elements in $\mathcal{L}$ intersect C at generic points.

Example 2.7. Let E be the exceptional curve of the blow up of O. Then E separates the pencil of jets of curve-germs corresponding to the order of vanishing at O.

Theorem 2.4 and Lemma 2.6 give the following characterization of curves that realize a given valuation.

Proposition 2.8. *Let ν be a divisorial discrete valuation centered at O and $\mathcal{L}_\nu$ be the corresponding pencil of jets of curve-germs. Let C be a curve on a birational lift of a neighborhood of O. Then C realizes ν iff C separates $\mathcal{L}_\nu$.*

Proof. Let U_0 be a neighborhood of O with analytic coordinates (u, v) and $\mathcal{L}_\nu$ be the pencil of jets of curve-germs at O corresponding to ν. We now construct a finite sequence of blow-ups $U_k \to U_{k-1} \to \cdots \to U_0$. Let U_1 be the blow-up of U_0 at O. Now assume that $i \geq 1$ and U_i has been constructed. Let E_i be the exceptional divisor of the blow up $U_i \to \bar{U}_{i-1}$. If the strict transforms of generic germs in $\mathcal{L}_\nu$ intersects E_i at generic points, then stop the sequence and set $k := i$. Otherwise the strict transforms of all germs in $\mathcal{L}_\nu$ passes through a unique point O_i of E_i. Let U_{i+1} be the blow-up of $\bar{U}_i$ at O_i. It follows from the classical theory of valuations centered at two dimensional local rings and the *third* definition of ν (in terms of intersection multiplicity) in Theorem 2.4 that the sequence of U_j's is finite and the 'last' exceptional divisor E_k realizes ν on U_k (see, *e.g.*, [CPR05, Section 2]).

Now we prove the proposition. Let C be a curve on a birational lift $\tilde{U}$ of a neighborhood U. Let $\phi\colon U_k \dashrightarrow \tilde{U}$ be the natural birational map (induced by the identification of a neighborhood of O). If C realizes ν, then Lemma 2.6 implies that ϕ maps generic points on E to generic points on C. Then it follows from the construction of E_k that the strict transforms of generic curve-germs of $\mathcal{L}_\nu$ intersect C at generic points. This proves ($\Rightarrow$) direction of the proposition. Conversely, if strict transforms of curve-germs in $\mathcal{L}_\nu$ intersects generic points on C_k, this forces $\phi|_{E_k}\colon E_k \dashrightarrow C$ to be generically finite-to-one. It follows from normality of $\tilde{U}$ and U_k that ϕ restricts to an isomorphism near generic points of E_k, which implies that C realizes ν, as required. □

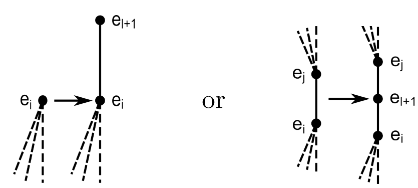
We now apply Theorem 2.4 to estimate the maximum number of singular points of compactifications of $\mathbb{C}^2$. The main argument is a basic property of the resolution graph of plane curve singularities (which was explained to me by Dmitry Kerner).

Proposition 2.9. *Let X be a compactification of $\mathbb{C}^2$. Assume that $X \setminus \mathbb{C}^2$ has N irreducible components. Then $|\operatorname{Sing}(X)| \leq 2N$, where* $\operatorname{Sing}(X)$ *is the set of singular points of X. In addition, if X is also a* minimal compactification *of $\mathbb{C}^2$ in the sense that none of the curves at infinity can be contracted, then $|\operatorname{Sing}(X)| \leq N + 1$.*

Proof. Let $X_0 := \mathbb{P}^2$ be the usual compactification of $\mathbb{C}^2$ via the embedding $(x, y) \mapsto ([x : y : 1])$ and let X be an arbitrary compactification of $\mathbb{C}^2$. Assume

$X \setminus \mathbb{C}^2$ has N components at infinity and let $\nu_1, \ldots, \nu_m$ be valuations *distinct* from $-\deg$ on $\mathbb{C}(x, y)$ corresponding to the curves at infinity on X. In particular, either $m = N$ or $m = N - 1$.

Applying the construction of the proof of Proposition 2.8 to the center of ν_1 on X_0, and then to the center of ν_2 on the resulting surface and so on, we may construct a finite sequence of blow-ups $X_s \to X_{s-1} \to \cdots \to X_0$ such that each ν_j, $1 \leq j \leq m$, is realized on X_s by one of the exceptional divisors. Let E_0 be the line at infinity on X_0 and for each l, $1 \leq l \leq s$, let E_l be the exceptional divisor from the blow up $X_l \to X_{l-1}$ and Γ_l be the 'augmented resolution graph', *i.e.*, Γ_l is the graph which has $l + 1$ vertices $e_0, \ldots, e_l$ and there is an edge between e_i and e_j iff the strict transforms of E_i and E_j intersect in X_l. Then it is straightforward to see (see *e.g.*, [Spi90b, Remark 5.5]) that for each l, there are only two possibilities for the transformation from Γ_l to Γ_{l+1}:



In particular,
(1) Γ_l is connected.
(2) E_l intersects at most two distinct E_j's in X_l; denote them as E_{l_0} and E_{l_1} (possibly $l_0 = l_1$).
(3) If $\tilde{\Gamma}$ is a connected component of $\Gamma_s \setminus \{e_l\}$ which does not contain e_{l_0} or e_{l_1}, then all the divisors of $\tilde{\Gamma}$ can be contracted to a *non-singular* point.

Observations (1) and (2) imply that for each $k \geq 1$, blowing down all but k of the E_j's from X_s introduces at most $2k$ singular points. Since X can be constructed from X_s by blowing down all but N of the E_j's, this proves the first assertion. Observation 3 and the minimality assumption on X then prove the last assertion. □

Now assume X is a *singular* compactification of $\mathbb{C}^2$. Then X_s constructed in the proof of Proposition 2.9 is a desingularization of X, and the corresponding resolution graph is determined by the resolution graph for the plane curve singularities whose branches are precisely the curvettes in $\mathcal{L}_{\nu_1}, \ldots, \mathcal{L}_{\nu_N}$. We now give a more detailed description of the resolution graph in the case that X is *primitive* (*i.e.*, $X \setminus \mathbb{C}^2$ is irreducible). Recall that the resolution graph of a plane curve singularity is uniquely determined by its *Puiseux pairs.* More precisely, assume that the Puiseux pairs of an irreducible plane curve singularity are $(q_1, p_1), \ldots, (q_m, p_m)$. Then the dual weighted graph (the weight of a vertex being the self-intersection number for the corresponding divisor) for the minimal resolution of the singularity is as in Figure 1, where we denoted the 'last exceptional divisor' by e^* and the left most vertex by e_1 (and left all other vertices untitled). The weights u_i^j and v_i^j satisfy: $u_i^0, v_i^0 \geq 1$ and $u_i^j, v_i^j \geq 2$

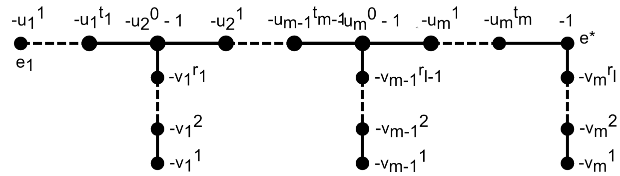


Figure 1

for $j > 0$, and are uniquely determined from the continued fractions (see, *e.g.*, [BK86, Section 8.4]):

(1)
$$\frac{p_i}{q'_i} = u_i^0 - \cfrac{1}{u_i^1 - \cfrac{1}{\ddots - \frac{1}{u_i^{t_i}}}}, \quad \frac{q'_i}{p_i} = v_i^0 - \cfrac{1}{v_i^1 - \cfrac{1}{\ddots - \frac{1}{v_i^{r_i}}}}, \quad \text{where } q'_i := q_i - q_{i-1}p_i.$$

Note that $(p_1, q'_1), \ldots, (p_l, q'_l)$ are the *Newton pairs* of the curve branch, *i.e.*, the Puiseux series of the branch is of the form:

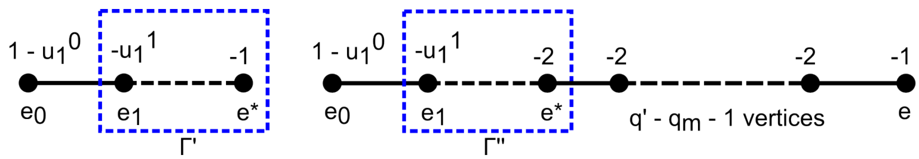
$$\psi(u) = \cdots + u^{\frac{q'_1}{p_1}}\Big(a'_1 + \cdots + u^{\frac{q'_2}{p_1 p_2}}\big(a'_2 + \cdots + u^{\frac{q'_3}{p_1 p_2 p_3}}(\cdots)\big)\Big).$$

Now let ν be the discrete valuation corresponding to the curve at infinity of X. Without loss of generality we may assume that ν is centered at $O := [1:0:0] \in \mathbb{P}^2$ and let $u := 1/x$ and $v := y/x$ be coordinates near O. Then an examination of the construction of X_s yields the following description of the 'augmented resolution graph' of the desingularization $X_s \to X$ (*i.e.*, Γ_s of the proof of Proposition 2.9) in terms of the resolution graph of generic curvettes in $\mathcal{L}_\nu$. Let $(q_1, p_1), \ldots, (q_m, p_m)$ be the Puiseux pairs (for Puiseux series expansion in u) of generic curvettes in $\mathcal{L}_\nu$, and let (q', p') be the *generic* formal Puiseux pair of $\mathcal{L}_\nu$. Then there are two possibilities: either $(q', p') = (q_m, p_m)$ or $p' = 1$ and $q' > q_m$.

PROPOSITION 2.10. *Let X be a primitive compactification of $\mathbb{C}^2$, Γ be the augmented resolution graph of $X_s \to X$ constructed in the proof of Proposition* 2.9 *and let (q_j, p_j), $1 \le j \le m$ and (q', p') be as above.*

(i) *If $p' = p_m$, then Γ is as in Figure* 2a*, where Γ' is as in figure* 1*. The vertex of Γ' corresponding to the strict transform of the curve $C_\infty := X \setminus \mathbb{C}^2$ is e^* (see Figure* 1*). In particular, one of the (at most two) singular points of X is a cyclic quotient singularity.*

(ii) *If $p' = 1$, then Γ is as in figure* 2b*, where Γ'' is the graph of Figure* 1 *with one change - namely the self-intersection number of e^* in Γ'' is -2. The vertex on Γ'' corresponding to the strict transform of C_∞ is e. In particular, X has one singular point.*



(a) (b)

Figure 2

REMARK 2.11. Note that in the case that $u_1^0 = 2$, the resolution of Proposition 2.10 is *not* minimal. We determine the minimal resolution of X in [Mon11a] by a case-by-case computation.

REMARK 2.12. Let Γ be a graph of one of the forms in Figure 2. Then Γ is a dual graph for a resolution of singularities of a primitive compactification of $\mathbb{C}^2$ iff the intersection product restricted to the connected components of $\Gamma \setminus \{v\}$ (where $v := e^*$ if Γ is as in Figure 2a and $v := e$ if Γ is as in Figure 2b) is negative definite. We give explicit formulæ in Corollary 3.7 to determine this negative definiteness and in Corollary 3.10 to determine if the result of blowing down corresponding exceptional divisors is a projective surface.

Before we proceed to state our main results in the next section, we introduce *degree-wise Puiseux series* corresponding to valuations *centered at infinity.*

DEFINITION 2.13. Let (x, y) be polynomial coordinates on $\mathbb{C}^2$ and let $\mathbb{C}^2 \hookrightarrow X := \mathbb{P}^2$ be the embedding given by $(x, y) \mapsto [x : y : 1]$. A *degree-wise Puiseux series* in x is a meromorphic Puiseux series in x^{-1}. For a discrete valuation ν on $\mathbb{C}(x, y)$, we say that ν is *centered at infinity* iff there exists $f \in \mathbb{C}[x, y]$ such that $\nu(f) < 0$. Assume ν is centered at infinity and $\nu \neq -\deg$, where deg is the usual degree of polynomials in (x, y)-coordinates. Then there is a unique point $O \in X \setminus \mathbb{C}^2$ such that ν is centered at O. After a linear change of coordinates if necessary we may assume that O has coordinates $[1 : 0 : 0]$. Then $u := 1/x$ and $v := y/x$ are coordinates near O and therefore generic curve germs in $\mathcal{L}_\nu$ have Puiseux series expansions of the form

$$v = \phi(u) + \xi u^r + (\text{higher order terms}), \text{ or equivalently,}$$
$$y = x\phi(1/x) + \xi x^{1-r} + (\text{lower order terms}),$$

for generic $\xi \in \mathbb{C}$. We say that $\psi_\nu(x, \xi) := x\phi(1/x) + \xi x^{1-r}$ is the *generic degree-wise Puiseux series* in x associated to ν. The *formal Puiseux pairs* of ψ_ν are precisely $(p_1 p_2 \cdots p_k - q_k, p_k)$, $1 \leq k \leq l + 1$, where (q_k, p_k), $1 \leq k \leq l + 1$, are the formal Puiseux pairs of $\mathcal{L}_\nu$ with respect to u. As in the case of $\mathcal{L}_\nu$, the last formal Puiseux pair of ψ_ν is called the *generic formal* Puiseux pair. It follows from Theorem 2.4 that for all $f \in \mathbb{C}(x,y)$,

$$\nu\big(f(x,y)\big) := -p \deg_x\big(f\big(x, \psi_\nu(x,\xi)\big)\big), \quad \text{where } p := p_1 \cdots p_{l+1}. \tag{2}$$

3. Main results.

PROPOSITION 3.1. *Let X_0 be a primitive projective compactification of $\mathbb{C}^2$ and ν be a discrete valuation on $\mathbb{C}(x,y)$ centered at infinity. Assume ν is not the valuation associated to the curve at infinity on X_0. Then there exists a normal projective compactification X of $\mathbb{C}^2$ and a morphism $\pi\colon X \to X_0$ such that*

(i) *$X \setminus \mathbb{C}^2$ consists of two irreducible curves D and E,*
(ii) *π contracts E to a point and restricts to an isomorphism on $X \setminus E$,*
(iii) *Let P be the point of intersection of D and E and $\mathcal{L}_\nu$ be the pencil of jets of curve-germs at infinity corresponding to ν. Then there is a one-to-one correspondence $\phi\colon E \setminus P \to \mathcal{L}_\nu$ such that for every $z \in E \setminus P$, the strict transform of a curve-germ C at infinity on $\mathbb{P}^2$ passes through z iff C is a representative of $\phi(z)$.*

REMARK 3.2. Assertion (iii) of Proposition 3.1 in particular implies that the strict transform on X of every curve-germ at infinity on X_0 which does not represent any element of $\mathcal{L}_\nu$ passes through P. This suggests that P corresponds to the *point at infinity* of the pencil $\mathcal{L}_\nu$.

PROOF. We only give a sketch of the proof in the case that $X_0 \cong \mathbb{P}^2$. The general case is proved in [Mon11a] by different arguments (which are much more elementary and 'constructive', but also much more lengthy). So assume $X_0 \cong \mathbb{P}^2$. Recall the construction of X_s from the proof of Proposition 2.9. Let X be constructed (as a normal *analytic* surface) from X_s by contracting all but one of the exceptional divisors. In particular, the projection $\pi\colon X_s \to X_0$ factors through X. Since X_0 is non-singular, it follows that every singularity of X is *sandwiched* (in the terminology of [Spi90a]), and therefore rational [Spi90a, Remark 1.15]. It follows that X is projective by a criterion of Artin [Art62, Theorem 2.3]. Assertions (i)—(iii) follow from the constructions of X_s and X. □

Let ν and X be as in Proposition 3.1. Then ν corresponds to a *primitive* compactification of $\mathbb{C}^2$ iff the curve D on X can be contracted to a (normal) point. It follows from Grauert and Artin's criterion that the latter is possible iff the self intersection number of D is negative. In [Mon11a] we compute the intersection matrix of D and E and consequently find an explicit criterion for determining if ν corresponds to a primitive compactification of $\mathbb{C}^2$.

DEFINITION 3.3. Let ν be a divisorial discrete valuation on $\mathbb{C}(x,y)$ centered at infinity and let $\psi_\nu(x,\xi) := \psi(x) + \xi x^r$ be the generic degree-wise Puiseux series corresponding to ν. Let

$$\Psi_\nu := \prod_{j=1}^{k} \big(y - \psi^{(j)}(x) - \xi x^r\big),$$

where $\psi^{(j)}$'s are the conjugates of ψ. Let Ψ^*_ν be the sum of all terms in Ψ_ν which are monomials in x, y and whose ν-value is less than or equal to the ν-value of every monomial for which the exponent of ξ is non-zero. Then $\Psi^*_\nu \in \mathbb{C}[x, x^{-1}, y]$. Let $s_\nu := -\nu(\Psi^*_\nu)$.

REMARK 3.4. If ν is the negative of usual degree in (x, y) coordinates, then $\psi_\nu = \xi x$, so that $\Psi_\nu = y - \xi x$, $s_\nu = 1$ and $\Psi^*_\nu = y$. Note that $\big\{V\big(\Psi_\nu(x, y, \xi)\big) : \xi \in \mathbb{C}\big\}$ is a family of curves in $\mathbb{P}^2$ which represents the pencil of jets of curve-germs at infinity corresponding to $-\deg$ and Ψ^*_ν is a representative of the jet corresponding to $\xi = 0$. The latter statement remains true for general ν: the curve C_ν defined by Ψ^*_ν in $\mathbb{P}^2$ is in a sense the 'simplest' *global curve* which represents the *zero*-th jet of ν at the center O of ν on the line at infinity of $\mathbb{P}^2$. Whether C_ν intersects the line at infinity at any other point except O turns out to be a deciding factor in determining if ν can be realized by a primitive *projective* compactification of $\mathbb{C}^2$ (see Theorem 3.8 and Lemma 3.19).

REMARK 3.5. It is straightforward (by induction on number of formal Puiseux pairs) to find an explicit formula for s_ν in terms of Puiseux pairs of ψ and r. Indeed, if the formal Puiseux pairs of ψ_ν are $(q_1, p_1), \ldots, (q_l, p_l), (q_{l+1}, p_{l+1})$, then

(3)
$$s_\nu = \sum_{j=1}^{l} \sum_{1 \le i_1 < \cdots < i_j \le l} (p_{i_j} - 1)(p_{i_{j-1}} - 1) \cdots (p_{i_1} - 1) q_{i_1} p_{i_1+1} \cdots p_{l+1} + q_{l+1}.$$

THEOREM 3.6. *ν corresponds to a primitive compactification of $\mathbb{C}^2$ iff $s_\nu > 0$.*

As an immediate application we find a characterization of graphs that arise from resolutions of singularities of primitive compactifications of $\mathbb{C}^2$.

COROLLARY 3.7. *Let Γ be a weighted graph as in Figure* 2*. Define $(q_1, p_1), \ldots, (q_m, p_m)$ from the continued fractions in* (1)*. If Γ is as in Figure* 2a*, then set $l := m - 1$. On the other hand, if Γ is as in Figure* 2b*, then define $l := m$, $p_{l+1} := 1$ and $q_{l+1} := q_l + 1 + l'$, where l' is the length of the 'right-most' string of (-2)-vertices starting from e^*. Finally, calculate s_Γ by the same formula of s_ν in equation* (3)*. Then Γ is the dual graph corresponding to a resolution of a primitive compactification of $\mathbb{C}^2$ iff $q_1 < p_1$ and $(p_1 \cdots p_l)^2 p_{l+1} > s_\Gamma$.* □

Theorem 3.6 gives a criterion to determine which discrete valuations correspond to primitive compactifications of $\mathbb{C}^2$, but gives no information as to determine if the compactification is algebraic or projective. Theorem 3.8, which is the main result of [Mon11b], gives an explicit answer.

Theorem 3.8. *Let ν be a divisorial discrete valuation on $\mathbb{C}(x, y)$ centered at infinity, and $\mathcal{L}_\nu$ be the pencil of jets of curve-germs at infinity corresponding to ν. Then the following are equivalent:*

(i) *ν corresponds to a primitive projective compactification of $\mathbb{C}^2$;*
(ii) *ν corresponds to a primitive algebraic compactification of $\mathbb{C}^2$;*
(iii) *one of the r-jets in $\mathcal{L}_\nu$ can be represented by a plane curve with one place at infinity;*
(iv) *the* zeroth *r-jet (*i.e.*, the one which corresponds to $\xi = 0$) in $\mathcal{L}_\nu$ can be represented by a plane curve with one place at infinity;*
(v) *$s_\nu > 0$ and $\Psi^*_\nu \in \mathbb{C}[x, y]$ (where Ψ^*_ν, s_ν are as in Definition* 3.3*).*

Remark 3.9. Theorems 3.6 and 3.8 imply the following characterization of primitive compactifications of $\mathbb{C}^2$ which are *not* algebraic: they correspond to ν such that $s_\nu > 0$ and $\Psi^*_\nu \notin \mathbb{C}[x, y]$ (*cf.* Example 4.1).

Theorem 3.8 gives a characterization of resolution graphs of primitive projective compactifications of $\mathbb{C}^2$. Indeed, let ν be a valuation on $\mathbb{C}(x, y)$ centered at infinity corresponding to a primitive projective compactification of $\mathbb{C}^2$ and let $\psi_\nu(x, \xi) := \psi(x) + \xi x^{\tilde{r}}$ be its generic degree-wise Puiseux series. Let $(\tilde{q}_1, p_1), \ldots, (\tilde{q}_l, p_l), (\tilde{q}_{l+1}, p_{l+1})$ be the formal Puiseux pairs of ψ_ν (in particular, the first l pairs are precisely the Puiseux pairs of $\psi(x)$). Let $s_0 := p_1 p_2 \cdots p_l$, and for each k, $1 \leq k \leq l$, define

$$s_k := \sum_{j=1}^{k-1} \sum_{1 \leq i_1 < \cdots < i_j \leq k-1} (p_{i_j} - 1)(p_{i_{j-1}} - 1) \cdots \cdots (p_{i_1} - 1)\tilde{q}_{i_1} p_{i_1+1} \cdots p_l + \tilde{q}_k p_{k+1} \cdots p_l \tag{4}$$

Then $p_{l+1}s_0, \ldots, p_{l+1}s_l$ and s_ν generate of the *semigroup of poles* at the place at infinity of the curve $C := V(\Psi^*_\nu)$ (it is a consequence of Theorem 3.8 that C is a planar curve with one place at infinity). Consequently, $s_0, \ldots, s_l$ satisfy the *semigroup condition* of Abhyankar–Moh [AM73]; in particular,

(S) $p_k s_k$ is in the semigroup generated by $s_0, \ldots, s_{k-1}$ for each k, $1 \leq k \leq l$.

Corollary 3.10. *Let Γ be a weighted graph as in Figure* 2*. Define $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$ as in Corollary* 3.7*. Set $\tilde{q}_k := p_1 \ldots p_k - q_k$, for $1 \leq k \leq l$, and define $s_0, \ldots, s_l$ as above. Then Γ is the dual graph corresponding to a resolution of a primitive projective compactification of $\mathbb{C}^2$ iff the conditions of Corollary* 3.7 *are satisfied and in addition* (S) *also holds.*

The proof of Theorem 3.8 is actually *constructive* in nature, in the sense that it gives the list of all equations defining primitive projective compactifications of $\mathbb{C}^2$ in terms of a fixed coordinate system (x, y) of $\mathbb{C}^2$, and consequently, the moduli space of primitive projective compactifications of $\mathbb{C}^2$. More precisely, we have the following corollary.

COROLLARY 3.11. *Let X be a primitive projective compactification of $\mathbb{C}^2$ and ν be the corresponding valuation on $\mathbb{C}(x,y)$ centered at infinity. Let $p_1,\dots,p_{l+1}$, $s_0,\dots,s_l$ be as above and set $s_{l+1} := s_\nu$. Then X is isomorphic to a subvariety of the weighted projective space $\mathbb{P}^{l+2}(1, p_{l+1}s_0, \dots, p_{l+1}s_l, s_{l+1})$ (with weighted homogeneous coordinates $[w : x : y_1 : \dots : y_{l+1}]$) defined by weighted homogeneous polynomials g_k, $1 \le k \le l$, of the form:*

$$(5)\quad g_k := w^{p_k s_k - s_{k+1}} y_{k+1} - \Big(y_k^{p_k} + a_k x^{\alpha_k} y_1^{\beta_{k,1}} \cdots y_{k-1}^{\beta_{k,k-1}} + \sum_{(\alpha,\vec{\beta})\in\Lambda_k} b_{\alpha\vec{\beta}} w^{m_{\alpha\vec{\beta}}} x^{\alpha} y_1^{\beta_1} \cdots y_k^{\beta_k} \Big),$$

where

(i) *$(\alpha_k, \beta_{k,1}, \dots, \beta_{k,k-1})$ is the unique collection of non-negative integers such that $\beta_{k,j} < p_j$ for each j, $1 \le j \le k-1$, and $\alpha_k p + \sum_{j=1}^{k-1} \beta_{k,j} s_j = p_k s_k$,*

(ii) *$\Lambda_k := \{(\alpha, \beta_1, \dots, \beta_k) \in \mathbb{N}^{k+1} : \beta_j < p_j$ for each j, $1 \le j \le k-1$, and $s_{k+1} < \alpha p + \sum_{j=1}^{k} \beta_j s_j < p_k s_k\}$,*

(iii) *$a_k \in \mathbb{C}^*$,*

(iv) *$b_{\alpha\vec{\beta}} \in \mathbb{C}$ and $m_{\alpha\vec{\beta}} := p_k s_k - (\alpha p + \sum_{j=1}^{k} \beta_j s_j)$ for each $(\alpha, \vec{\beta}) \in \Lambda_k$.*

Conversely, every collection of equations $g_1, \dots, g_l$ defined by equation (5) *defines a primitive projective compactification of $\mathbb{C}^2$ with formal (degree-wise) Puiseux pairs $(\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})$.*

REMARK 3.12. In particular, every primitive projective compactification of $\mathbb{C}^2$ is a weighted homogeneous complete intersection.

REMARK 3.13. Note the similarity of (5) with the description in [FS02, Theorem 7] of plane curves with one place at infinity. In fact, given Theorem 3.8, Corollary 3.11 and [FS02, Theorem 7] are equivalent. It follows that the structures of moduli spaces of primitive projective compactifications of $\mathbb{C}^2$ and planar curves with one place at infinity are also similar (*cf.* Corollary 3.14 and [FS02, Corollary 1]).

To calculate the moduli space of primitive projective compactifications of $\mathbb{C}^2$ up to isomorphism, it is necessary to put the generic degree-wise Puiseux series $\psi_\nu(x,\xi) := \psi(x) + \xi x^{\tilde{r}}$ of the corresponding valuation ν in a 'normal form'. Let the formal Puiseux pairs of $\psi(x)$ be $(\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})$. Then it is straightforward to see (*e.g.*, by induction on number of Puiseux pairs) that after a change of coordinates if necessary we may assume that one of the two following (mutually exclusive) conditions hold:

(N1) $\psi \equiv 0$ (so that $l = 0$) and $p_1 \ge \tilde{q}_1 \ge 1$, or

(N2) $l \ge 1$ and $p_1 > \tilde{q}_1 > 1$.

We say that the formal Puiseux pairs of ν are in the *normal form* if either (N1) or (N2) holds; in which case, we will simply say that the normal form of ν is $\big((\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})\big)$. Using the fact that the polynomial automorphisms of

$\mathbb{C}^2$ are *tame* [Jun42], it is not hard to see that the normal form is unique for a given ν.

COROLLARY 3.14. *Let X be a primitive projective compactification of $\mathbb{C}^2$ and let $(\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})$ be the normal form of the valuation ν on $\mathbb{C}(x,y)$ corresponding to the curve at infinity on X. Let $s_0, \dots, s_{l+1}$ be as in Corollary 3.11. Then X is isomorphic to a subvariety of $\mathbb{P}^{l+2}(1, p_{l+1}s_0, \dots, p_{l+1}s_l, s_{l+1})$ defined by a* unique *collection of weighted homogeneous polynomials $(g_1, \dots, g_l)$ given by equation (5). In particular, the moduli space $\mathcal{X}_{\vec{q},\vec{p}}$ of primitive projective compactifications of $\mathbb{C}^2$ with normal form $(\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})$ is isomorphic to $(\mathbb{C}^*)^l \times \mathbb{C}^{\lambda_l}$ where $\lambda_l := \sum_{k=1}^{l} |\Lambda_k|$ (Λ_k, $1 \leq k \leq l$, being as in Corollary 3.11).*

REMARK 3.15. In particular, the only primitive compactifications of $\mathbb{C}^2$ for which the normal form has only one pair (*i.e.*, $l = 0$) are weighted projective surfaces $\mathbb{P}^2(1, p, q)$.

REMARK 3.16. Using Theorem 3.6, the moduli space $\mathcal{X}_{an}$ of primitive *analytic* compactifications of $\mathbb{C}^2$ can be computed in the same way in terms of coefficients of the associated degree-wise Puiseux series. In particular, the moduli space $\mathcal{X}_{an,\vec{q},\vec{p}}$ for a given normal form $(\tilde{q}_1, p_1), \dots, (\tilde{q}_{l+1}, p_{l+1})$ has the form $(\mathbb{C}^*)^l \times \mathbb{C}^{\lambda_l^*}$ for some $\lambda_l^* \geq 0$ (which counts, in the same way as λ_l, the cardinality of some set of indices) and $\mathcal{X}_{\vec{q},\vec{p}}$ is a closed subvariety of $\mathcal{X}_{an,\vec{q},\vec{p}}$ (it is of course possible that $\mathcal{X}_{\vec{q},\vec{p}}$ is empty or equals $\mathcal{X}_{an,\vec{q},\vec{p}}$).

As a final application of Theorem 3.8, we give an effective answer to Question 1.1. More precisely, we answer the following question: given the degree-wise Puiseux series for a discrete valuation ν centered at infinity, is it possible to *effectively* determine if $\nu < 0$ on *all* non-constant polynomials?

PROPOSITION 3.17. *Let ν be a discrete valuation on $\mathbb{C}(x,y)$ centered at infinity. Let s_ν and Ψ_ν^* be as in Theorem 3.8. Then*

(1) *ν is non-positive on $\mathbb{C}[x,y]$ iff $s_\nu \geq 0$;*
(2) *ν is negative on all non-constant $\mathbb{C}[x,y]$ iff either*
   (a) *$s_\nu > 0$, or*
   (b) *$s_\nu = 0$ and $\Psi_\nu^* \notin \mathbb{C}[x,y]$.*

REMARK 3.18. In particular, the answer to Question 1.1 is negative: there are divisorial discrete valuations which are negative on all non-constant polynomials in $\mathbb{C}[x,y]$ but do not correspond to any primitive compactifications of $\mathbb{C}^2$, namely those which satisfy (2b) of Proposition 3.17. For example, let ν be the valuation corresponding to degree-wise Puiseux series $\psi_\nu := x^{2/5} + x^{-1} + \xi x^{-8/5}$ (in particular, as equation (2) implies, $\nu\big(f(x,y)\big) := -5\deg_x\big(f(x,\psi_\nu)\big)$ for all

$f \in \mathbb{C}(x,y)$). Then

$$\Psi_\nu := \prod_{k=1}^{5}(y - \zeta^{2k}x^{2/5} - x^{-1} + \xi x^{-8/5}) \quad (\text{where } \zeta := e^{2\pi i/5})$$
$$= y^5 - x^2 - 5x^{-1}y^4 - 5\xi x^{-\frac{8}{5}}y^4 + (\text{higher order terms})$$
$$(\text{where 'higher order terms' means higher order w.r.t. } \nu)$$
$$s_\nu := -\nu(x^{-\frac{8}{5}}y^4) = 0, \quad \Psi^*_\nu := y^5 - x^2 - 5x^{-1}y^4 \notin \mathbb{C}[x,y].$$

It follows that ν satisfies assertion (2b) of Proposition 3.17 and therefore does not correspond to any primitive compactification of $\mathbb{C}^2$.

We end this section with stating a lemma (essentially about factorization of polynomials) which is the main technical ingredient in the proof of Theorem 3.8 and seems interesting in its own right. Let $O \in \mathbb{P}^2$ and (u,v) be *linear* affine coordinates at O (that is $u = 0$ and $v = 0$ are lines in $\mathbb{P}^2$).

Lemma 3.19. *Let $\mathcal{J}$ be an (u,ω)-jet at O for some $\omega > 0$ and let $L \subseteq \mathbb{P}^2$ be the line $u = 0$. Then the following are equivalent:*
(1) *There is an algebraic curve $C \subseteq \mathbb{P}^2$ such that C intersects L only at O and each branch of C at O belongs to $\mathcal{J}$.*
(2) *There is an algebraic curve $C \subseteq \mathbb{P}^2$ such that C intersects L only at O, C is unibranch at O, and the branch of C at O belongs to $\mathcal{J}$.*

4. Examples, remarks and conjectures.

Example 4.1. Consider the families of curve germs at infinity with degree-wise Puiseux expansions:

$$y = \psi_1(x,\xi) := x^{2/5} + \xi x^{-7/5}, \quad y = \psi_2(x,\xi) := x^{2/5} + x^{-1} + \xi x^{-7/5},$$

where ξ varies in $\mathbb{C}$. These correspond to families of curve-germs at $O := [1:0:0] \in \mathbb{P}^2$ (the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ being as $(x,y) \mapsto [x:y:1]$). Indeed, in coordinates given by $u := 1/x$ and $v := y/x$, the equations $y = \psi_j(x,\xi)$, $1 \le j \le 2$ translate to the following:

$$v = u^{3/5} + \xi u^{12/5}, \quad v = u^{3/5} + u^2 + \xi u^{12/5}.$$

Theorem 3.6 and Proposition 2.10 imply that for each j, $1 \le j \le 2$, the family of (12/5)-jets at O associated to $y = \psi_j(x,\xi)$ corresponds to a primitive analytic compactification X_j of $\mathbb{C}^2$ with the 'augmented resolution graph' (as in Proposition 2.10) as in Figure 3. (The dual graph for the *minimal resolution* of X_j's are constructed by removing e and e_1 from Figure 3 and changing the weight of the vertex next to e_1 from -3 to -2.) In particular, both X_1 and X_2 are analytic compactifications of $\mathbb{C}^2$ with a unique singular point and the same

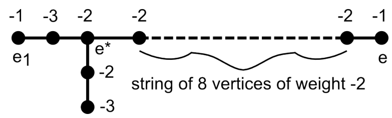


Figure 3

resolution graph. It is straightforward to see that $\Psi_1^* = y^5 - x^2$, which implies via Theorem 3.8 that X_1 is projective. A direct application of Corollary 3.11 shows that X_1 is the hypersurface in $\mathbb{P}^3(1,5,2,1)$ (with weighted homogeneous coordinates $[w : x : y : z]$) defined by the equation $w^9z = x^2 - y^5$. On the other hand, $\Psi_2^* = y^5 - x^2 - 5x^{-1}y^4$, and therefore X_2 is *not* even algebraic!

REMARK 4.2. It is easy to find (in essentially the same way as in Example 4.1) examples of non-algebraic compactifications of $\mathbb{C}^2$ with arbitrarily many irreducible curves at infinity.

Theorem 3.8 and Lemma 3.19 give a criterion (which also follows from Zariski–Fujita Theorem [Laz04, Remark 2.1.32]) for *algebraically* contracting a chain of rational curves on a non-singular compactification of $\mathbb{C}^2$. More general forms of this question was considered in [Art62], [Art66] and [Sch00]. The criterion ($\diamondsuit$) is more in the spirit of [Sch00] which gives a characterization for contractible curves in terms of complementary Weil divisors.

($\diamondsuit$) Let X be a normal compactification of $\mathbb{C}^2$, $D \subseteq X \setminus \mathbb{C}^2$ and E be the union of the irreducible curves in $X \setminus \mathbb{C}^2$ excluding the components of D. Assume that E is *analytically* contractible (*i.e.*, the intersection matrix of components of E is negative definite) and D is irreducible. Then E is algebraically contractible iff there is a curve $C \subseteq \mathbb{C}^2$ such that $\bar{C} \cap D = \varnothing$.

Another equivalent formulation of ($\diamondsuit$) is

($\clubsuit$) A primitive normal compactification X of $\mathbb{C}^2$ is algebraic iff for every point $P \in X$, there is a curve $C \subseteq \mathbb{C}^2$ such that $P \notin \bar{C}$.

CONJECTURE 4.3. Statement ($\diamondsuit$) remains true without the assumption that D is irreducible; equivalently, ($\clubsuit$) remains true for all (*i.e.*, not necessarily primitive) normal compactifications of $\mathbb{C}^2$. In addition, if E of statement ($\diamondsuit$) is algebraically contractible, then it is in fact *projectively* contractible (*i.e.*, the contraction of E on X produces a projective surface).

QUESTION 4.4. More generally, we ask: does ($\clubsuit$) remains valid for X being arbitrary rational normal Moishezon surfaces (with $\mathbb{C}^2$ being replaced by an open subset $U \subseteq X$ which is isomorphic to an open affine subset of $\mathbb{C}^2$)?

If Conjecture 4.3 is indeed true, then determining the algebraicity of a normal compactification of $\mathbb{C}^2$ will involve answering the following question:

(♠) Given finitely many (not necessarily distinct) points $P_1, \dots P_m$ on a line $L \subseteq \mathbb{P}^2$ and curve-jets $\mathcal{J}_k$ centered at P_k, $1 \leq k \leq m$, when does there exist an algebraic curve $C \subseteq \mathbb{P}^2$ such that $C \cap L \subseteq \{P_1, \dots, P_m\}$ and each branch of C at a point in $C \cap L$ belongs to one of the $\mathcal{J}_k$'s?

QUESTION 4.5. In particular, we ask: can (♠) be answered *effectively*?

REMARK 4.6. Lemma 3.19 and assertion (v) of Theorem 3.8 give an answer to Question 4.5 in the case that $m = 1$.

REFERENCES

[**AM73**] Shreeram S. Abhyankar and Tzuong Tsieng Moh, *Newton–Puiseux expansion and generalized Tschirnhausen transformation. I, II.* J. Reine Angew. Math. **260** (1973), 47–83; *ibid.*, **261** (1973), 29–54.

[**AM75**] ———, *Embeddings of the line in the plane.* J. Reine Angew. Math. **276** (1975), 148–166.

[**Art62**] Michael Artin, *Some numerical criteria for contractability of curves on algebraic surfaces.* Amer. J. Math. **84** (1962), 485–496.

[**Art66**] ———, *On isolated rational singularities of surfaces.* Amer. J. Math. **88** (1966), 129–136.

[**BDP81**] Lawrence Brenton, Daniel Drucker, and Geert C. E. Prins, *Graph theoretic techniques in algebraic geometry. I. The extended Dynkin diagram $\bar{E}_8$ and minimal singular compactifications of* $\mathbf{C}^2$. Ann. of Math. Stud. **100**, Princeton Univ. Press, Princeton, N.J., 1981, 47–63.

[**Ber75**] D. N. Bernstein, *The number of roots of a system of equations.* Funkcional. Anal. i Priložen. **9** (1975), 1–4.

[**BK86**] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves.* Birkhäuser Verlag, Basel, 1986 (translated from the German by John Stillwell).

[**Bre73**] Lawrence Brenton, *A note on compactifications of* $\mathbf{C}^2$. Math. Ann. **206** (1973), 303–310.

[**Bre80**] ———, *On singular complex surfaces with negative canonical bundle, with applications to singular compactifications of* $\mathbf{C}^2$ *and to 3-dimensional rational singularities.* Math. Ann. **248** (1980), 117–124.

[**CPR05**] Antonio Campillo, Olivier Piltant, and Ana J. Reguera, *Cones of curves and of line bundles "at infinity".* J. Algebra **293** (2005), 513–542.

[**FJ04**] Charles Favre and Mattias Jonsson, *The valuative tree.* Lecture Notes in Math. **1853**, Springer-Verlag, Berlin, 2004.

[**FJ07**] Charles Favre and Mattias Jonsson, *Eigenvaluations.* Ann. Sci. École Norm. Sup. **40** (2007), 309–349.

[**FJ11**] ———, *Dynamical compactifications of* $\mathbf{C}^2$. Ann. of Math. **173** (2011), 211–248.

[**FS02**] Mitsushi Fujimoto and Masakazu Suzuki, *Construction of affine plane curves with one place at infinity.* Osaka J. Math. **39** (2002), 1005–1027.

[**Fur97**] Mikio Furushima, *On minimal compactifications of* $\mathbf{C}^2$. Math. Nachr. **186** (1997), 115–129.

[**Gan79**] Richard Ganong, *On plane curves with one place at infinity.* J. Reine Angew. Math. **307/308** (1979), 173–193.

[**Gra62**] Hans Grauert, *Über Modifikationen und exzeptionelle analytische Mengen.* Math. Ann. **146** (1962), 331–368.

[**Jun42**] Heinrich W. E. Jung, *Über ganze birationale Transformationen der Ebene.* J. Reine Angew. Math. **184** (1942), 161–174.

[**Koj01**] Hideo Kojima, *Minimal singular compactifications of the affine plane.* Nihonkai Math. J. **12** (2001), 165–195.

[**KT09**] Hideo Kojima and Takeshi Takahashi, *Notes on minimal compactifications of the affine plane.* Ann. Mat. Pura Appl. **188** (2009), 153–169.

[**Kus76**] A. G. Kushnirenko, *Newton polyhedra and Bezout's theorem.* Funkcional. Anal. i Priložen. **10** (1976), 3, 82–83.

[**Laz04**] Robert Lazarsfeld, *Positivity in algebraic geometry. I.* Ergeb. Math. Grenzgeb. (3) **49**, Springer-Verlag, Berlin, 2004.

[**MF11**] Julio José Moyano-Fernández, *Curvettes and clusters of infinitely near points.* Rev. Mat. Complut. **24** (2011), 439–463.

[**Mon10**] Pinaki Mondal, *Towards a Bezout-type theory of affine varieties.* PhD thesis, University of Toronto, 2010.

[**Mon11a**] ———, *Primitive normal compactifications of the affine plane I.* arXiv:1110.6905, 2011.

[**Mon11b**] ———, *Primitive normal compactifications of the affine plane II.* arXiv:1110.6914, 2011.

[**Mor72**] James A. Morrow, *Compactifications of* $\mathbf{C}^2$. Bull. Amer. Math. Soc. **78** (1972), 813–816.

[**MZ88**] M. Miyanishi and D.-Q. Zhang, *Gorenstein log del Pezzo surfaces of rank one.* J. Algebra **118** (1988), 63–84.

[**Nag58**] Masayoshi Nagata, *Existence theorems for nonprojective complete algebraic varieties.* Illinois J. Math. **2** (1958), 490–498.

[**NO97**] Yuji Nakazawa and Mutsuo Oka, *Smooth plane curves with one place at infinity.* J. Math. Soc. Japan **49** (1997), 663–687.

[**Oht01**] Tomoaki Ohta, *Normal hypersurfaces as a compactification of* $\mathbb{C}^2$. Kyushu J. Math. **55** (2001), 165–181.

[**Oka98**] Mutsuo Oka, *Moduli space of smooth affine curves of a given genus with one place at infinity.* In: Singularities (Oberwolfach, 1996), Progr. Math. **162**, Birkhäuser, Basel, 1998, 409–434.

[**Rus80**] Peter Russell, *Hamburger–Noether expansions and approximate roots of polynomials.* Manuscripta Math. **31** (1980), 25–95.

[**Sch00**] Stefan Schröer, *On contractible curves on normal surfaces.* J. Reine Angew. Math. **524** (2000), 1–15.

[**Spi90a**] Mark Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations.* Ann. of Math. (2) **131** (1990), 411–491.

[**Spi90b**] ———, *Valuations in function fields of surfaces.* Amer. J. Math. **112** (1990), 107–156.

[**Suz99**] Masakazu Suzuki, *Affine plane curves with one place at infinity.* Ann. Inst. Fourier (Grenoble) **49** (1999), 375–404.

University of Toronto, Toronto, ON
e-mail: pinaki@math.toronto.edu