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New Solutions for Non-smooth Value Functions

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Abstract. In this paper, we provide strong solutions to the stochastic optimization problem when the value function is not necessarily smooth.

Résumé. Des solutions fortes du problème d'optimisation stochastique sont fournies dans des cas où la fonction d'utilité indirecte n'est pas forcément dérivable.

1. **Introduction.** Viscosity solutions were introduced by Crandall and Lions [4] as weak solutions to the Hamilton–Jacobi–Bellman (HJB) partial differential equations (PDE). Later constrained viscosity solutions were introduced (see, for example, Soner [7]). Viscosity solutions have been extensively used in stochastic analysis (see [7] again, among many others). In this note, we provide strong solutions to the stochastic optimization problem when the value function is not necessarily smooth in the spatial argument. We achieve this by bypassing the traditional HJB PDE and providing an alternative PDE that has a unique classical solution.

2. **The model.** As an example, we use the standard investment model (see, for example, Alghalith [2] among many others). Similar to previous models, we assume that the investor starts with initial wealth $X_t \equiv x$ at time t and he/she invests this wealth on a risky portfolio (a risky asset or a combination of risky assets) π_t such as a stock, and a risk-free asset B_t such as a bond or bank account. The portfolio process π_s is defined as the amount of money held in the risky asset(s) at a time $s \geq t$. At the initial time t (and the subsequent times), the investor chooses the optimal portfolio and deposits the remainder of his/her wealth in the bank. The choice of the portfolio depends on the information available at time s, which is represented by the filtration $\{\mathcal{F}_s\}_{t \leq s \leq T}$.

We consider a risky asset and risk-free asset; we also assume that the parameters of the model depend on a random external economic factor Y_s such as the unemployment rate. In doing so, we adopt a two-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, \mathcal{F}_s\}_{t \leq s \leq T}$ on the probability space $(\Omega, \mathcal{F}_s, P)$. The

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price of the risky asset is given by $S_s(s, W_{1s}) = S_t e^{(\mu(Y_s)-.5\sigma^2(Y_s))(s-t)+\sigma(Y_s)W_{1s}}$. Applying Ito's rule[1] to this expression, we obtain the dynamics of the risky asset price

(1) $$dS_s = S_s\{\mu(Y_s)ds + \sigma(Y_s)dW_{1s}\},$$

where $\mu(Y_s)$ and $\sigma(Y_s)$ are the rate of return and the volatility, respectively. The risk-free asset price process is given by $\bar{S} = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(R)$ is the rate of return. The dynamics of $\bar{S}$ are given by

(2) $$d\bar{S} = r\bar{S}dt.$$

The dynamics of the economic factor process are given by

(3) $$dY_s = g(Y_s)ds + \rho dW_{1s} + \sqrt{1-\rho^2}dW_{2s}, \quad Y_t = y,$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $g(Y_s) \in C^1(R)$ with a bounded derivative.

The change in wealth is equal to the number of shares of the risky asset times the change in its price, plus the number of shares of the risk-free asset times the change in its price:

(4) $$dX = \frac{\pi}{S}ds + \frac{X-\pi}{\bar{S}}d\bar{S}.$$

Substituting (1) and (2) into (4), we obtain

(5) $$dX_s = \{r(Y_s)X_s^\pi + \big(\mu(Y_s) - r(Y_s)\big)\pi_s\}ds + \pi_s\sigma(Y_s)dW_{1s}.$$

Thus, the terminal wealth (at time T) can be expressed in the integral form

(6) $$X_T^\pi = x + \int_t^T \{r(Y_s)X_s^\pi + \big(\mu(Y_s) - r(Y_s)\big)\pi_s\}\, ds + \int_t^T \pi_s\sigma(Y_s)\, dW_{1s}, \quad t \le s \le T.$$

The trading strategy $\pi_s \in \mathcal{A}(x, y)$ is admissible ($X_s > 0$ with $\int_t^T \pi_s^2\, ds < \infty$). Following Alghalith [1, p. 913], we rewrite (6) as (simply decomposing the integral)

(7) $$X_T^\pi = x + r_t x + (\mu_t - r_t)\pi_t + \pi_t\sigma_t W_{1t} + \int_{\hat{t}>t}^T \{r_s X_s^\pi + (\mu_s - r_s)\pi_s\}\, ds + \int_{\hat{t}>t}^T \pi_s\sigma_s dW_{1s},$$

[1] According to Ito's rule, the dynamics of a stochastic process $G(s, W_s)$ are given by

$$dG = G_s ds + G_W dW_s + \frac{1}{2}G_{WW}(dW_s)^2 = G_s ds + G_W dW_s + \frac{1}{2}G_{WW} ds,$$

where the subscripts denote partial derivatives.

where we suppressed the notational dependence on Y_s. We can also rewrite (6) as

$$X_T^\pi = x + \beta + r_t x + \alpha(\mu_t - r_t)\pi_t + \pi_t \sigma_t W_{1t} + \int_{\hat{t}>t}^T \{r_s X_s^\pi + (\mu_s - r_s)\pi_s\}\, ds + \int_{\hat{t}>t}^T \pi_s \sigma_s \, dW_{1s}, \tag{8}$$

where α is a shift parameter with initial value equal to one, β is a shift parameter with initial value equal to zero (see [3, p. 443]). Note that, at the initial values of the shift parameters, (8) is equal to (7).

The investor's objective is to maximize the expected utility[2] (satisfaction) of the terminal wealth (that is, the investor chooses the portfolio that maximizes his/her utility)

$$\begin{aligned} V(t,x,y,\alpha,\beta) &= \mathrm{Sup}_\pi E[U(X_T^\pi) \mid \mathcal{F}_t] \\ &= E\bigg[U\bigg(x + \beta + r_t x + \alpha(\mu_t - r_t)\pi_t^* + \pi_t^* \sigma_t W_{1t} \\ &\qquad + \int_{\hat{t}>t}^T \{r_s X_s^\pi + (\mu_s - r_s)\pi_s^*\}\, ds + \int_{\hat{t}>t}^T \pi_s^* \sigma_s dW_{1s}\bigg) \,\bigg|\, \mathcal{F}_t\bigg], \end{aligned} \tag{9}$$

where $V(\,\cdot\,)$ is the value function (the maximum utility at time t, given that the initial wealth is equal to x), $U(\,\cdot\,)$ is a continuous, bounded and strictly concave utility function. The optimal value is indicated with an asterisk. Also $V(\,\cdot\,)$ is (separately) continuously differentiable with respect to the shift parameters. This is because the derivative of V with respect to the shift parameter is a function of the parameters (that is, $\partial V(\,\cdot\,)/\partial\alpha \equiv V_\alpha(\alpha,\beta,\,\cdot\,)$ and $\partial V(\,\cdot\,)/\partial\beta \equiv V_\beta(\alpha,\beta,\,\cdot\,)$) and by construction a function is differentiable with respect to the shift parameter (see, for example, [3, p. 444]). This guarantees a smooth solution to the PDE in the next section.

3. The solutions. Differentiating both sides of (9) with respect to α and β, respectively, we obtain

$$V_\alpha(\,\cdot\,) = (\mu_t - r_t)\pi_t^* E[U'(X_T^\pi) \mid \mathcal{F}_t], \tag{10}$$

$$V_\beta(\,\cdot\,) = E[U'(X_T^\pi) \mid \mathcal{F}_t], \tag{11}$$

where the subscripts denote partial derivatives; dividing (10) by (11) yields

$$\frac{V_\alpha(\,\cdot\,)}{V_\beta(\,\cdot\,)} = (\mu_t - r_t)\pi_t^*. \tag{12}$$

[2]The utility function measures the investor's level of satisfaction/happiness obtained from the wealth; a larger wealth yields a higher level of satisfaction ($\partial U(X)/\partial X > 0$).

Now we define $c \equiv -(\mu_t - r_t)\pi_t^*$ and rewrite (12) as

$$V_\alpha(\,\cdot\,) + cV_\beta(\,\cdot\,) = 0. \tag{13}$$

This is a well-known simple PDE. It is established that it has a solution of the form $V = F(\beta - c\alpha)$, given the initial condition $V(\beta, 0) = F(\beta)$. Clearly, the solution is classical since V is continuously differentiable in the parameters (see, for example, Pinchover and Rubinstein [6, p. 3]), even if V is not smooth in the spatial argument.

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