C. R. Math. Rep. Acad. Sci. Canada Vol. **34** (4) 2012, pp. 101–104

On a Problem of Gromov about Generalizing the Alexandrov–Fenchel Inequality

Yuri Burda

Presented by Pierre Milman, FRSC

Abstract. In this note we give an answer to a question about mixed volumes asked by Gromov in "Convex Sets and Kahler Manifolds". For the reader's convenience we recall the definitions and some of the properties of mixed volumes and mixed discriminants.

Résumé. Dans cette note, nous donnions une réponse à une question sur les volumes mixtes posées par M. Gromov dans "Convex Sets and Kahler Manifolds". Pour la commodité du lecteur, nous rappelons les définitions et certaines des propriétés de volumes mixtes et discriminants mixte.

1. Mixed volumes and mixed discriminants. By a theorem of Minkowski the volume of a positive linear combination $\lambda_1 A_1 + \cdots + \lambda_k A_k$ of k convex bodies in $\mathbf{R}^n$ is a homogeneous polynomial of degree n in λ's:

$$V(\lambda_1 A_1 + \cdots + \lambda_k A_k) = \sum_{\substack{I \subset \mathbf{Z}_+^k \\ |I|=n}} \binom{n}{I} V_I \lambda^I.$$

The coefficient V_I for $I = \{i_1, \ldots, i_k\}$ is called the mixed volume

$$V(\underbrace{A_1, \ldots, A_1}_{i_1}, \ldots, \underbrace{A_k, \ldots, A_k}_{i_k})$$

of the bodies $A_1, \ldots, A_1, \ldots, A_k, \ldots, A_k$.

For example the mixed volume of n copies of a convex body in $\mathbf{R}^n$ is the usual volume of the body. The mixed volume of $n-1$ copies of a convex body A and one copy of the unit ball is the $n-1$-dimensional volume $V_{n-1}(\partial A)$ of the boundary of A divided by n. See for example [BZ88] for introduction and further examples.

For a set $A_1, \ldots, A_k$ of $n \times n$ real symmetric matrices the mixed discriminant

$$\det(\underbrace{A_1, \ldots, A_1}_{i_1}, \ldots, \underbrace{A_k, \ldots, A_k}_{i_k})$$

Received by the editors on October 3, 2011.
AMS Subject Classification: 52A39.

is the coefficient $D_{\{i_1,\dots,i_k\}}$ in the expansion

$$\det(\lambda_1 A_1 + \cdots + \lambda_k A_k) = \sum_{\substack{I\subset \mathbf{Z}_+^k \\ |I|=n}} \binom{n}{I} D_I \lambda^I.$$

1.1. Example. If each A_i is a box $A_i = [0, a_{i1}] \times \cdots \times [0, a_{in}]$ then the mixed volume of the bodies $A_1, \dots, A_n$ is the permanent of the matrix $(a_{ij})_{i,j=1}^n$ divided by $n!$.

Similarly if A_i is the diagonal matrix with diagonal entries $a_{i1}, \dots, a_{in}$ then the mixed discriminant of the matrices $A_1, \dots, A_n$ is the permanent of the matrix $(a_{ij})_{i,j=1}^n$ divided by $n!$.

2. Alexandrov–Fenchel inequality and its analogue in linear context.

The Alexandrov–Fenchel inequality states that for a set of n bodies $A_1, \dots, A_n$ in $\mathbf{R}^n$

$$V(A_1, A_2, A_3, \dots, A_n)^2 \geq V(A_1, A_1, A_3, \dots, A_n) V(A_2, A_2, A_3, \dots, A_n).$$

This inequality generalizes many known inequalities for convex bodies, such as the isoperimetric inequality,

$$\frac{V_n(A)^{1/n}}{V_{n-1}(\partial A)^{1/(n-1)}} \leq \frac{V_n(B^n)^{1/n}}{V_{n-1}(\partial B^n)^{1/(n-1)}},$$

the Brunn–Minkowski inequality,

$$V(A+B)^{1/n} \geq V(A)^{1/n} + V(B)^{1/n},$$

and many others.

The Alexandrov–Fenchel inequality was first announced by Fenchel [Fen36] and proved by Alexandrov in [Ale37] and [Ale38]. A simpler proof has been recently found in [McM93] and [Tim99].

This inequality is important not only because it is one of the most general inequalities known about convex bodies, but also because of its relations with algebraic geometry. Khovanskii and Tessier have found (see [Kho], [Tei79] and the discussion in [Gro90]) that Alexandrov inequality can be derived from its algebro-geometric analogue

$$[D_1, D_2, D_3, \dots, D_n]^2 \geq [D_1, D_1, D_3, \dots, D_n][D_2, D_2, D_3, \dots, D_n]$$

where D_i are ample divisors in a smooth irreducible algebraic variety and $[-, \dots, -]$ is the intersection index of divisors. This algebro-geometric analogue can be derived as a consequence of Hodge index theorem on a smooth irreducible algebraic surface.

The Alexandrov–Fenchel inequality has an analogue for real positive-definite symmetric matrices $A_1, \dots, A_n$:

$$\det(A_1, A_2, A_3, \dots, A_n)^2 \geq \det(A_1, A_1, A_3, \dots, A_n) \det(A_2, A_2, A_3, \dots, A_n).$$

This inequality was first proved in [Ale38] and subsequently given a simpler proof in [Kho84].

Even the most simple cases of these inequalities are extremely useful. For instance when the bodies are boxes with parallel sides (or when the symmetric matrices are diagonal), the inequality on permanents implied by Alexandrov-Fenchel inequality has been used by Falikman [Fal81] in 1979 and by Egorychev in 1980 [Ego81] to prove a conjecture of van der Waerden that has been open since 1926. Namely they proved that the minimal value of permanent of a doubly stochastic $n \times n$ matrix is attained on the matrix all of whose values are equal to $1/n$.

3. A negative answer to Gromov's question. Alexandrov–Fenchel inequalities can be equivalently formulated in terms of the function $f(I) = \log(V_I)$ (or $f(I) = \log(D_I)$) on the discrete simplex $\{I \subset \mathbf{Z}^n_+, |I| = n\}$, where V_I (or D_I) are the mixed volumes (or mixed discriminants) appearing in the definition of the mixed volume (or discriminant) above. Namely, assuming in addition that $\log 0 = -\infty$, Alexandrov–Fenchel inequality says that the function f is concave on any segment in the discrete simplex that is parallel to one of the sides.

In [Gro90] Gromov asked whether it was true that the function f is concave on the discrete simplex.

In case $n = 3$ this generalization amounts to the inequality

$$V(A_1, A_2, A_3)^3 \geq V(A_1, A_1, A_2)V(A_2, A_2, A_3)V(A_3, A_3, A_1).$$

We claim that this inequality fails even when the bodies are boxes with sides parallel to the axes.

Namely we can take $A_1 = [0, 1] \times [0, 1] \times \{0\}$, $A_2 = [0, 1] \times \{0\} \times [0, 5]$ and $A_3 = \{0\} \times [0, 1/3] \times [0, 1]$.

Then $V(A_1, A_2, A_3) = 4/9$, $V(A_1, A_1, A_2) = 5/3$, $V(A_2, A_2, A_3) = 5/9$, $V(A_3, A_3, A_1) = 1/9$ and $(4/9)^3 < 5/3 \cdot 5/9 \cdot 1/9$.

ACKNOWLEDGEMENTS. I would like to thank Askold Khovanskii for introducing me to the theory of mixed volumes and Dmitry Faifman and Yevgeny Liokumovich for inspiring discussions.

REFERENCES

[**Ale37**] A. D. Alexandrov, *Zur Theorie der Gemischten Volumina von konvexen Körpern II; neue ungleichungen zwischen den gemischten Volumina und ihren Anwendungen.* Math. Sbomik NS **2** (1937), 1205–1238.

[**Ale38**] ———, *Zur Theorie der Gemischten Volumina von konvexen Körpern IV; die gemischten Diskriminanten und die gemischten Volumina*. Math. Sbomik NS **3** (1938), 227–251.

[**BZ88**] Y. D. Burago and V. A. Zalgaller, *Geometric Inequalities.* Grundlehren Math. Wiss., Springer, Berlin, 1988.

[**Ego81**] G. P. Egorychev, *The solution of van der Waerden's problem for permanents.* Adv. in Math. **42** (1981), 299–305.

[**Fal81**] D. I. Falikman, *Proof of the van der Waerden conjecture regarding the permanent of a doubly stochastic matrix.* (English) Math. Notes **29** (1981), 475–479.

[**Fen36**] W. Fenchel, *Généralisations du théorème de Brunn et Minkowski concernant les corps convexes.* C. R. Acad. Sci. Paris **203** (1936), 764–766.

[**Gro90**] M. Gromov, *Convex sets and Kähler manifolds.* In: Advances in Differential Geometry and Topology (ed. F. Tricerri), World Sci. Publ., Singapore, 1990, 1–38.

[**Kho**] A. G. Khovanskii, *Algebra and mixed volumes.* Appendix 3 in [BZ88].

[**Kho84**] ———, *Analogues of the Aleksandrov–Fenchel inequalities for hyperbolic forms.* In: Soviet Math. Dokl. **29** (1984), 710–713.

[**McM93**] P. McMullen, *On simple polytopes.* Invent. Math. **113** (1993), 419–444.

[**Tei79**] B. Teissier, *Du théoreme de l'index de Hodge aux inégalités isopérimétriques.* C. R. Acad. Sci. Paris Ser. A–B **288**, 1979.

[**Tim99**] V. A. Timorin, *An analogue of the Hodge–Riemann relations for simple convex polytopes.* Russian Math. Surveys **54** (1999), 381.

Department of Mathematics, University of Toronto, Toronto, ON
e-mail: yburda@math.toronto.edu