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# Bezout-type Theorems for the Affine Plane

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Abstract. This article is mainly an announcement of some of the results from the article *General Bezout-type theorems.* We set up a framework for Bezout-type theorems for general affine varieties and apply it to study the "Bezout problem" on the affine plane. In particular, for a class of compactifications of the affine plane we compute the intersection numbers of the curves at infinity in terms of valuations at infinity, and generalize the Bernstein–Kushnirenko non-degeneracy criterion to the case of weighted degrees in possibly *different* systems of coordinates.

Résumé. Cet article est principalement une annonce de quelques-uns des résultats de l'article *General Bezout-type theorems.* Nous construisons un système pour les théorèmes de type Bezout pour les variétés affines générales, et nous l'appliquons pour étudier le "Bezout problème" sur le plan affine. En particulier, pour une classe de compactifications du plan affine, nous calculons le nombre d'intersections des, courbes à l'infini en termes de valorisations à l'infini, et nous généralisons le critère de non-dégénérescence de Bernstein–Kushnirenko au cas de degrés pondérés dans les systèmes peut-être différents de coordonnées.

1. **Introduction.** This work started as a project to understand 'affine Bezout type' theorems, *i.e.*, those which estimate the number of solutions of a system of equations on an *affine* variety. The most famous theorems of this kind, apart from the Bezout theorem, are perhaps those of Kushnirenko [Kus76] and Bernstein [Ber75]. Both these theorems count numbers of solutions on $(\mathbb{C}^*)^n$ of systems of $n$ Laurent polynomials (*i.e.*, elements of $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$). Recall that the *Newton polytope* of a Laurent polynomial $f$ is the convex hull in $\mathbb{R}^n$ of the exponents of the monomials that appear in $f$. Bernstein's theorem, which is a generalization of Kushnirenko's theorem, is the following.

Theorem 1.1 (Bernstein [Ber75]). *The number of isolated solutions in $(\mathbb{C}^*)^n$ of Laurent polynomials $f_1, \ldots, f_n$ is bounded above by $n!$ times the mixed volume of the Newton polytopes of $f_1, \ldots, f_n$. The bound is exact iff $f_1, \ldots, f_n$ satisfy*

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*the following:*

(∗) *For each weighted degree $\omega$ on $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, there is no common zero in $(\mathbb{C}^*)^n$ of the leading forms of $f_1, \ldots, f_n$ with respect to $\omega$.*

After the seminal work of A. Khovanskii (see, *e.g.*, [Kho77] and [Kho78]) which *explained* and generalized this result, the bound in Bernstein's theorem came to be known as the *BKK bound*. There have been numerous works which generalize this theorem and provide formulae for the number of solutions of systems of equations on affine varieties, the systems being *generic* in some suitable sense (see, *e.g.*, [HS95], [HS97], [LW96], [Roj94], [Roj99], [RW96]).

Notation 1.2. Throughout this article the base field, unless otherwise stated, will be an algebraically closed field $\mathbb{K}$ of arbitrary characteristic, and if $X$ is an affine algebraic variety over $\mathbb{K}$, then $\mathbb{K}[X]$ will denote the ring of regular functions on $X$.

The point of departure of this article is that a basic ingredient of Kushnirenko and Bernstein's theorems (and several of their generalizations) is the following (easy to verify) property of compactifications of affine varieties.

Lemma 1.3. *Let $\bar{X}_1, \ldots, \bar{X}_n$ be compactifications of an affine variety $X$ of dimension $n$ and for each $j$, $1 \leq j \leq n$, assume that there exists a base-point free divisor $D_j$ on $\bar{X}_j$ with support in $\bar{X}_j \setminus X$. Define $\bar{X}$ to be closure of the image of $X$ under the* diagonal map *$X \hookrightarrow \bar{X}_1 \times \cdots \times \bar{X}_n$. Then for generic $f_1 \ldots, f_n \in \mathbb{K}[X]$ such that $f_j$ is the restriction of a global section of (the line bundle $\mathcal{O}_{\bar{X}_j}(D_j)$ corresponding to) $D_j$ for each $j$, $1 \leq j \leq n$, the number of solutions (counted with appropriate multiplicity) on $X$ of $f_1, \ldots, f_n$ is precisely the intersection number of $\pi_1^*(D_1), \ldots, \pi_n^*(D_n)$ on $\bar{X}$, where $\pi_j \colon \bar{X} \to \bar{X}_j$ is the natural projection in the $j$-th coordinate, $1 \leq j \leq n$.* □

Remark-Definition 1.4. Lemma 1.3 was used in [KK08, Section 7] to define *intersection numbers* of linear systems. Indeed, in the terminology of [KK08], if $L_j$ is the vector subspace of $\mathbb{K}[X]$ consisting of global sections of $\mathcal{O}_{\bar{X}_j}(D_j)$, $1 \leq j \leq n$, then the number in the conclusion of Lemma 1.3 is precisely the *intersection number* $[L_1, \ldots, L_n]$ (which is by definition the number of solutions on $X$ of generic $f_1, \ldots, f_n$ with $f_j \in L_j$, $1 \leq j \leq n$). However, we mostly use Lemma 1.3 in the special case that $D_j$'s are also ample, or more precisely, the compactifications $\bar{X}_j$'s correspond to *subdegrees* (see Definition 2.4). In this form Lemma 1.3 implicitly appeared in [Mon10b, Theorem 3.3.2].

Let $X$ be as in Lemma 1.3 and let $f := (f_1, \ldots, f_n) \colon X \to \mathbb{K}^n$ be a finite-to-one map. We consider two counting problems associated to $f$: the *Bezout problem* for $f$ is to count the number of solutions (with appropriate multiplicity) on $X$ of $f_1, \ldots, f_n$, whereas the *generic Bezout problem* for $f$ is to count the number of solutions on $X$ of $f_1 - a_1, \dots, f_n - a_n$ for generic $a := (a_1, \dots, a_n) \in \mathbb{K}^n$. Note that for a given $f$, the answer to the generic Bezout problem is always greater than or equal to the answer to the Bezout problem, and they are equal iff $f$ is proper in a neighborhood of $f^{-1}(0)$. Lemma 1.3 suggests one approach to solve the Bezout problem (resp. generic Bezout problem) for $f$:

Step 1: Find linear systems $L_1, \dots, L_n$ of regular functions on $X$ such that $f_1, \dots, f_n$ are *non-degenerate* (resp. *generically non-degenerate*) with respect to $L_1, \dots, L_n$ in the sense that the number of solutions on $X$ of $f_1, \dots, f_n$ (resp. $f_1 - a_1, \dots, f_n - a_n$ for generic $(a_1, \dots, a_n) \in \mathbb{K}^n$) is precisely $[L_1, \dots, L_n]$.

Step 2: Realize $L_j$'s as (the restrictions onto $X$ of) $H^0\big(\mathcal{O}_{\bar{X}_j}(D_j)\big)$ for base-point free divisors $D_j$ supported at infinity on some compactifications $\bar{X}_j$ of $X$, and

Step 3: Compute $[L_1, \dots, L_n]$ using intersection theory on $\bar{X}$ of Lemma 1.3.

In order to have any success in this approach, the next step is necessary:

Step 4: Give an explicit calculus (*e.g.*, similar to the criterion $(*)$) to verify when $f_1, \dots, f_n$ are non-degenerate with respect to $L_1, \dots, L_n$ in the sense of 1.

EXAMPLE 1.5. Consider $f := (x^2 - y^3, x - y^2)\colon \mathbb{K}^2 \to \mathbb{K}^2$. Then the bound on $|f^{-1}(0)|$ (counted with multiplicity) given by Bezout's theorem is 6, but a straightforward calculation shows that the actual number is 4. To compute this number via the approach outlined above, consider the weighted projective space $\bar{X} := \mathbb{P}^2(2, 1, 1)$ with weighted homogeneous coordinates $[u : v : w]$. The identification of $x$ with $u/w^2$ and $y$ with $v/w$ shows that $\bar{X}$ is a compactification of $\mathbb{K}^2$. Let $D$ be the $\mathbb{Q}$-Cartier divisor at infinity corresponding to the (irreducible) curve $C := \bar{X} \setminus \mathbb{K}^2$. Then $D$ is ample (in fact $2D$ is very ample) and Steps 1 and 2 are realized with $\bar{X}_1 = \bar{X}_2 := \bar{X}$ and $L_1 := \mathcal{O}_{\bar{X}}(4D)$, $L_2 := \mathcal{O}_{\bar{X}}(2D)$. It follows then from the intersection theory of the weighted homogeneous spaces that the number of points in $f^{-1}(0)$ is $\frac{\mathrm{wt}(f_1)\,\mathrm{wt}(f_2)}{wt(x)wt(y)} = 4$ (where wt denotes the weighted degree of a polynomial corresponding to the weights 2 for $x$ and 1 for $y$). Note that in this case the solution to both versions of the Bezout problem is the same, since $f$ is proper. Moreover, Step 4 can also be realized in this case, *i.e.*, there is an explicit criterion for determining non-degeneracy with respect to (linear systems of divisors at infinity on a weighted projective space determined by) a given weighted degree, namely: $f_1, \dots, f_n$ are non-degenerate iff the leading weighted homogeneous forms of $f_j$'s have only one common solution, namely the origin (see *e.g.*, [Dam99]).

It turns out that for the Bezout problem, Steps 1 and 2 can be achieved at least for $\mathbb{K} = \mathbb{C}$ and $n = 2$ (Proposition 3.5). It seems very plausible (but we are unable to prove) that the same approach should work for affine varieties of arbitrary dimensions. We can however show that the corresponding statement for the *generic* Bezout problem is true in arbitrary dimensions—see Proposition 3.3 for a simple proof (the proof being essentially a remark of the referee). Therefore, at least for solving the generic Bezout problem via the above approach, the essential difficulty seems to be in carrying out Steps 3 and 4 for general systems of polynomials. Our results in this article give partial results in this direction for $X = \mathbb{K}^2$. A more precise summary of the results is as follows.

(1) For $X = \mathbb{K}^2$ (or more generally, an affine surface with no non-constant invertible regular functions), we compute intersection numbers of curves at infinity on a class of compactifications on $X$. More precisely, given *primitive* projective compactifications $\bar{X}_1, \ldots, \bar{X}_k$ of $X$ (a compactification $Z$ of $X$ is *primitive* if $Z \setminus X$ is irreducible), we consider the normalization $\bar{X}$ of the closure of the diagonal embedding of $X$ into $\bar{X}_1 \times \cdots \times \bar{X}_k$ and compute the intersection matrix of the curves at infinity on $\bar{X}$ in terms of the *linking number* of the associated valuations (Lemma 3.6).

(2) For $\mathbb{K} := \mathbb{C}$ and $X := \mathbb{C}^2$, we give in Proposition 3.15 a natural generalization of the BKK non-degeneracy criterion $(*)$ for linear systems defined by collections of weighted degrees in possibly *different* systems of coordinates.

In the next section we briefly outline the theory of compactifications corresponding to *subdegrees* (as developed in [Mon10b] and [Mon10a])—we only describe as much as it is necessary to state Proposition 2.8 (which is necessary for Lemma 3.6). In Section 3 we state and sketch the proof of main results. In Section 4 we work out in details an example of a polynomial system which is *BKK degenerate* (*i.e.*, does not satisfy $(*)$), but for which our methods can count the exact number of solutions.

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## 2. Background: semidegrees and subdegrees.

Definition 2.1. Let $X$ be an irreducible affine variety over $\mathbb{K}$. A map $\delta\colon \mathbb{K}[X] \setminus \{0\} \to \mathbb{Z}$ is called a *degree-like function* if

(i) $\delta(f+g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{K}[X]$, with $<$ in the preceding inequality implying $\delta(f) = \delta(g)$.

(ii) $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in \mathbb{K}[X]$.

Every degree-like function $\delta$ on $\mathbb{K}[X]$ defines an *ascending filtration* $\mathcal{F}^\delta := \{F_d^\delta\}_{d \geq 0}$ on $\mathbb{K}[X]$, where $F_d^\delta := \{f \in \mathbb{K}[X] : \delta(f) \leq d\}$. Let $t$ be an indeterminate and define

$$\mathbb{K}[X]^\delta := \bigoplus_{d\geq 0} F_d^\delta t^d, \quad \operatorname{gr}\mathbb{K}[X]^\delta := \bigoplus_{d\geq 0} F_d^\delta t^d / F_{d-1}^\delta t^{d-1}.$$

We say that $\delta$ is *finitely-generated* if $\mathbb{K}[X]^\delta$ is a finitely-generated algebra over $\mathbb{K}$ and that $\delta$ is *projective* if in addition $F_0^\delta = \mathbb{K}$. The motivation for the terminology comes from the following.

PROPOSITION 2.2 ([Mon10b, Proposition 1.1.2]). *If $\delta$ is a projective degree-like function, then $\bar{X}^\delta := \operatorname{Proj}\mathbb{K}[X]^\delta$ is a projective compactification of $X$. The* hypersurface at infinity *$\bar{X}^\delta_\infty := \bar{X}^\delta \setminus X$ is the zero set of the $\mathbb{Q}$-Cartier divisor defined by $t$ and is isomorphic to $\operatorname{Proj}\operatorname{gr}\mathbb{K}[X]^\delta$. Conversely, if $\bar{X}$ is any projective compactification of $X$ such that $\bar{X} \setminus X$ is the support of an effective ample divisor, then there is a projective degree-like function $\delta$ on $\mathbb{K}[X]$ such that $\bar{X}^\delta \cong \bar{X}$.*

REMARK 2.3. That $t$ in general defines a $\mathbb{Q}$-Cartier divisor (as opposed to a usual Cartier divisor) can be seen from Example 1.5, where the role of $t$ is played by $w$.

DEFINITION 2.4. A degree-like function $\delta$ is called a *semidegree* if it always satisfies property (ii) with an equality, and $\delta$ is called a *subdegree* if it is the maximum of finitely many semidegrees.

REMARK-EXAMPLE 2.5. Our motivation for considering subdegrees and semidegrees come from *toric geometry*. Indeed, assume $\mathcal{P}$ is an $n$-dimensional convex polytope in $\mathbb{R}^n$ that contains the origin in the interior and whose vertices have rational coordinates. Define $\delta_\mathcal{P}\colon \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \setminus \{0\} \to \mathbb{Q}$ such that

$$\delta_\mathcal{P}\Big(\sum a_\alpha x^\alpha\Big) := \inf\{r \in \mathbb{R} : \alpha \in r\mathcal{P} \text{ for all } \alpha \in \mathbb{Z}^n \text{ such that } a_\alpha \neq 0\}.$$

Pick any positive integer $e$ such that $e\delta_\mathcal{P}$ is integer-valued. Then it is straightforward to see that $e\delta_\mathcal{P}$ is a subdegree which is the maximum of weighted degrees determined by the *facets* (*i.e.*, codimension one faces) of $\mathcal{P}$. The corresponding compactification of $X := (\mathbb{K}^*)^n$ is precisely the *toric variety* corresponding to $\mathcal{P}$. It follows from the theory of toric varieties that there is a one-to-one correspondence between the facets of $\mathcal{P}$ and irreducible components of the complement of the torus in $X_\mathcal{P}$. The following result extends this property to the case of arbitrary subdegrees.

THEOREM 2.6 (*cf.* [Mon10b, Theorem 2.2.1]). *Let $\delta$ be a projective subdegree on the coordinate ring of an irreducible affine variety $X$. Then*

(i) *$\delta$ has a unique minimal presentation as the maximum of finitely many semidegrees.*

(ii) *The semidegrees in the minimal presentation of $\delta$ are (up to integer multiples) precisely the orders of pole along the irreducible components of the hypersurface at infinity.*

REMARK 2.7. The content of assertion (i) of Theorem 2.6 is precisely the uniqueness of the minimal presentation. The assertion remains true even if $\delta$ is *not* projective. In the case that $\mathbb{K}[X]^\delta$ is Noetherian, the argument needed to prove assertion (i) gives a characterization of subdegrees, namely: $\delta$ is a subdegree iff $\delta(f^k) = k\delta(f)$ for all $f \in \mathbb{K}[X]$ and $k \geq 0$. Assertion (ii) implies in particular that the local ring of $\bar{X}^\delta$ at each irreducible component of $\bar{X}^\delta_\infty$ is a discrete valuation ring (which in turn implies that the codimension of $\operatorname{Sing}(\bar{X}^\delta) \cap \bar{X}^\delta_\infty$ in $\bar{X}^\delta$ is at least two).

Given any degree-like function $\delta$ on $\mathbb{K}[X]$, the *divisor at infinity* $D^\delta_\infty$ on $\bar{X}^\delta$ is the $\mathbb{Q}$-Cartier divisor defined by $t$ of Proposition 2.2. It is straightforward to see that $D^\delta_\infty$ is ample and its support is $\bar{X}^\delta_\infty$. We later use the following formula for the pull-back of $D^\delta_\infty$ under a regular map.

PROPOSITION 2.8 ([Mon10a, Proposition 4.26]). *Let $X$ be an affine variety, $\delta$ be a finitely generated non-negative subdegree on $\mathbb{K}[X]$ and $\phi\colon Z \to \bar{X}^\delta$ be a dominant morphism. Then*

$$\phi^*(D^\delta_\infty) = \sum_W l^\phi_\infty(\delta, \operatorname{pole}_W)[W], \tag{1}$$

*where the sum is over codimension one irreducible subvarieties $W$ of $Z$, and for each such $W$, the function* $\operatorname{pole}_W$ *is the negative of the* order $\operatorname{ord}_W$ of vanishing *along $W$* ($\operatorname{ord}_W$ *being defined as in* [Ful98, Section 1.2]) *and*

$$l^\phi_\infty(\delta, \operatorname{pole}_W) := \max\Big\{\frac{\operatorname{pole}_W\big(\phi^*(f)\big)}{\delta(f)} : f \in \mathbb{K}[X], \delta(f) > 0\Big\}.$$

REMARK 2.9. Identity (1) in particular implies that $l^\phi_\infty(\delta, \operatorname{pole}_W)$ exists. Note also that the sum in identity (1) is indeed finite, since $l^\phi_\infty(\delta, \operatorname{pole}_W) = 0$ for all codimension one irreducible subvariety $W$ of $Z$ such that $W \cap \phi^{-1}(X) \neq \varnothing$.

REMARK-DEFINITION 2.10. Given a Krull local ring $\mathfrak{o}$, a valuation $\omega$ on $\mathfrak{o}$ and a height one prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, the *linking number* [Huc70] of $\omega$ and the valuation $\nu_\mathfrak{p}$ on $\mathfrak{o}$ corresponding to $\mathfrak{p}$ is

$$l(\omega, \nu_\mathfrak{p}) := \inf_{f \in \mathfrak{p}, f \neq 0} \frac{\omega(f)}{\nu_\mathfrak{p}(f)}.$$

The linking number was introduced in [Sam59] in connection with defining the pull-back of Weil divisors under a birational regular mapping. More precisely, if $\phi\colon Z \to Y$ is a birational map and $\omega$ (resp. $\nu_\mathfrak{p}$) is the order of vanishing along a codimension one subvariety $W$ of $Z$ (resp. $V$ of $Y$), then $l(\omega, \nu_{\mathfrak{p}})$ is a 'candidate' for the coefficient of $[W]$ in $\phi^*([V])$. Now assume $Y := \bar{X}^\delta$ for a *semidegree* $\delta$ (*i.e.*, $Y \setminus X$ is irreducible) and $V := Y \setminus X$. Then $\delta = -\nu_{\mathfrak{p}}$ and $\mathrm{pole}_W = -\omega$ and Proposition 2.8 implies that

$$l^{\phi}_{\infty}(\delta, \mathrm{pole}_W) = l(\omega, \nu_{\mathfrak{p}}).$$

It is perhaps instructive that $l(\omega, \nu_{\mathfrak{p}})$ and $l^{\phi}_{\infty}(\delta, \mathrm{pole}_W)$ measure the *same* quantity, but the "inf" of the local computation is replaced by a "sup" in the global computation. We call $l^{\phi}_{\infty}(\delta, \mathrm{pole}_W)$ the *linking number at infinity* (*relative to* $\phi$) of $\eta$ and $\mathrm{pole}_W$. In the case that $\phi$ is birational (so that $\phi^*$ is an identity on the function field), we simply write $l_\infty(\delta, \mathrm{pole}_W)$ for $l^{\phi}_{\infty}(\delta, \mathrm{pole}_W)$.

## 3. Main results.

DEFINITION 3.1. Let $X$ be an affine variety of dimension $n$ and $\bar{X}$ be a compactification of $X$. We say that $\bar{X}$ *preserves the intersection of subvarieties* $V_1, \ldots, V_k$ *of* $X$ *at* $\infty$ if $\bar{V}_1 \cap \cdots \cap \bar{V}_k \cap X_\infty = \varnothing$, where $X_\infty := \bar{X} \setminus X$ is the set of 'points at infinity' and $\bar{V}_j$ is the closure of $V_j$ in $\bar{X}$ for every $j$. If $f := (f_1, \ldots, f_n) \colon X \to \mathbb{K}^n$ is a regular map, we associate two *linear systems* $L_f$ and $L_f^{\mathrm{gen}}$ to $f$, which are defined as follows: $L_f$ (resp. $L_f^{\mathrm{gen}}$) is the $n$ (resp. $n+1$) dimensional $\mathbb{K}$-vector subspace of $\mathbb{K}[X]$ generated by $f_1, \ldots, f_n$ (resp. $1, f_1, \ldots, f_n$). We say that $\bar{X}$ *preserves generic intersections of* $L_f$ *(resp.* $L_f^{\mathrm{gen}}$*)* if $\bar{X}$ preserves the intersection at $\infty$ of $V(g_1), \ldots, V(g_n)$ for generic $g_1, \ldots, g_n \in L_f$ (resp. $L_f^{\mathrm{gen}}$).

REMARK 3.2. The motivation for considering $L_f$ and $L_f^{\mathrm{gen}}$ is that the $n$-th self intersection number (in the sense of Remark-Definition 1.4) of $L_f$ (resp. $L_f^{\mathrm{gen}}$) is precisely the solution for the Bezout problem (resp. generic Bezout problem).

The following proposition shows that it is possible to carry out Steps 1 and 2 of Introduction for the generic Bezout problem.

PROPOSITION 3.3. *Let* $X$ *be an affine variety of dimension* $n$ *and* $f \colon X \to \mathbb{K}^n$ *be a finite-to-one polynomial map.*

(i) *There is a projective compactification* $\bar{X}$ *of* $X$ *which preserves generic intersections of* $L_f^{\mathrm{gen}}$.

(ii) *If* $X$ *is normal, then there is a projective compactification* $\bar{X}$ *of* $X$ *and a base-point free ample divisor* $D$ *on* $\bar{X}$ *with support in* $\bar{X} \setminus X$ *such that* $L_f^{\mathrm{gen}} \subseteq H^0(\mathcal{O}_{\bar{X}}(D))|_X$ *and the number of solutions on* $X$ *of* $n$ *generic elements of* $L_f^{\mathrm{gen}}$ *is precisely* $D^n$.

PROOF (FOLLOWING A REMARK OF THE REFEREE). Let $\bar{X}_0$ be any projective compactification of $X$. Set $\bar{X}$ to be the closure of the graph of $f$ in $\bar{X} \times \mathbb{P}^n(\mathbb{K})$. It is straightforward to see that $\bar{X}$ preserves generic intersections of $L_f^{\mathrm{gen}}$, which proves assertion (i).

Now assume that $X$ is normal. Let $\bar{X}'$ be the normalization of $\bar{X}$ and $\pi\colon \bar{X}' \to \mathbb{P}^n(\mathbb{K})$ be the natural projection. Then $\pi$ extends $f$. Let $\bar{X}' \xrightarrow{\phi_1} Z \xrightarrow{\phi_2} \mathbb{P}^n(\mathbb{K})$ be the Stein factorization of $\pi$. Since $\pi$ is finite-to-one on $X$ and $Z$ is normal, it follows by Zariski's main theorem that $\phi_1|_X$ is an isomorphism. The divisors of poles of generic elements of $L_f^{\text{gen}}$ on $Z$ are *identical* (as Weil divisors)—denote them by $D_f$. Then $D_f$ is the pullback of the hyperplane divisor at infinity on $\mathbb{P}^n(\mathbb{K})$ and therefore ample and base-point free. Consequently, setting $\bar{X} := Z$ and $D := D_f$ satisfies assertion (ii). □

Remark 3.4. We can show (by a much more complicated proof) that assertion (i) remains true if $f$ is only assumed to be dominant. But we do not know of any applications of this fact.

We now show that for $n = 2$, Proposition 3.3 remains *almost true* with $L_f^{\text{gen}}$ replaced by $L_f$. Recall that a Cartier divisor $D$ on a variety $Z$ is called *semi-ample* if $\mathcal{O}_Z(mD)$ is generated by global sections for some positive integer $m$.

Proposition 3.5. *Let $X$ be an affine algebraic surface over $\mathbb{K}$ and let $f := (f_1, f_2)\colon X \to \mathbb{K}^2$ be a regular map such that $f^{-1}(0)$ is finite. Then there is a compactification $\bar{X}$ of $X$ such that*

(i) *$\bar{X}$ preserves generic intersections of $L_f$.*
(ii) *There is a semi-ample divisor $D$ on $\bar{X}$ with support in $\bar{X} \setminus X$ such that $L_f \subseteq H^0\big(\mathcal{O}_{\bar{X}}(D)\big)|_X$ and the number of solutions on $X$ of two generic elements of $L_f$ is precisely $D^2$.*

Sketch of a Proof. Without loss of generality, we may assume that $X = \mathbb{K}^2$, since the assertion for general surfaces follows from Noether normalization. Embed $X \hookrightarrow \bar{X}_0 := \mathbb{P}^2$ via $(x, y) \mapsto [1 : x : y]$. For each $g \in L_f$, let us denote by $C_g$ (resp. $\bar{C}_g^0$) the curve on $X$ defined by $g$ (resp. the closure of $C_g$ in $\bar{X}_0$). There are finitely many points $P_1, \dots, P_k \in \bar{X}_0 \setminus X$ such that each $P_j$ belongs to $\bar{C}_g^0$ for more than one $g \in L_f$. For each $j$, there exists a finite sequence of blow-ups centered at $P_j$ and points in infinitesimal neighborhoods of $P_j$ such that generic pairs of curves $\bar{C}_g^0$ meeting at $P_j$ get *separated* in the resulting surface. It follows that the surface $\bar{X}$ resulting from performing these sequences of blow-ups for all $P_j$'s satisfies assertion (i). For each $g \in L_f$, let $\bar{C}_g$ be the closure of $C_g$ in $\bar{X}$. Fix a generic $g \in L_f$ and set $D := [\bar{C}_g] - \operatorname{div}(g)$, where $\operatorname{div}(g)$ denotes the divisor of $g$ on $\bar{X}$. Then $L_f \subseteq H^0\big(\mathcal{O}_{\bar{X}}(D)\big)|_X$, and therefore, by the assumption on $f$, the base locus of $\mathcal{O}_{\bar{X}}(D)$ is finite. Consequently, by the Zariski–Fujita theorem (see, *e.g.*, [Laz04, Remark 2.1.32]), $D$ is semi-ample. Since $\bar{X}$ satisfies assertion (i), this completes the proof of assertion (ii). □

Propositions 3.3 and 3.5 state that the number of solutions of polynomial systems on an affine surface can be computed in terms of intersection numbers of divisors supported at infinity on compactifications of the surface. Our next result (Lemma 3.6) computes the intersection theory at infinity on a certain class of compactifications of $\mathbb{K}^2$.

Let $\bar{X}_1, \dots, \bar{X}_k$ be non-isomorphic normal projective compactifications of $X := \mathbb{K}^2$ such that the complement $C_j$ of $X$ in each $\bar{X}_j$ is an irreducible curve. Let $\delta_j : \mathbb{K}[x,y] \to \mathbb{N}$ be the order of pole of polynomials along $C_j$. In the terminology of Section 2, each $\delta_j$ is a *projective semidegree*. Let $\bar{X}$ be the normalization of the closure in $\bar{X}_1 \times \cdots \times \bar{X}_k$ of the image of $X$ under the *diagonal* mapping. The complement of $X$ in $\bar{X}$ has precisely $k$ irreducible components $\tilde{C}_1, \dots, \tilde{C}_k$, where $\tilde{C}_j$ is the unique curve in $\bar{X} \setminus X$ which maps *onto* $C_j$ under the natural projection $\pi_j : \bar{X} \to \bar{X}_j$. The intersection numbers $(\tilde{C}_i, \tilde{C}_j)$ are well defined (according to Mumford's intersection theory for normal complete surfaces [Mum61]). Define three $k \times k$ matrices $\mathcal{L}, \mathcal{I}, \mathcal{D}$ as follows:

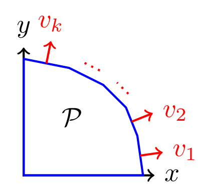
$\mathcal{L} :=$ matrix of linking numbers at infinity of $\delta_j$'s with the $(i,j)$-th entry being $l_\infty(\delta_i, \delta_j)$,
$\mathcal{D} :=$ the diagonal matrix with the $i$-th diagonal entry being $(C_i, C_i)$,
$\mathcal{I} :=$ matrix of intersection numbers $\tilde{C}_j$'s with the $(i,j)$-th entry being $(\tilde{C}_i, \tilde{C}_j)$.

LEMMA 3.6. *$\mathcal{L}\mathcal{I} = \mathcal{D}$.*

PROOF. Pick $i, j$, $1 \le i, j \le k$. If $i \ne j$, then $\pi_i(\tilde{C}_j)$ is a point and therefore $\left(\pi_i^*(C_i), \tilde{C}_j\right) = 0$. Since intersection numbers are preserved by pullbacks under birational morphisms, it follows that $(C_i, C_i) = \left(\pi_i^*(C_i), \pi_i^*(C_i)\right) = \left(\pi_i^*(C_i), \tilde{C}_i\right)$. The lemma now follows from Proposition 2.8 which implies that $\pi_i^*(C_i) = \sum_{l=1}^k l_{ik}\tilde{C}_k$. □

As an application of Lemma 3.6 we compute the intersection theory of a class of toric compactifications of $\mathbb{K}^2$. Let $\mathcal{P}$ be a convex rational polygon in $\mathbb{R}^2$ with

- one vertex at the origin,
- two edges along the axes, and
- the other edges being line segments with *negative* rational slopes.




List the non-axis edges of $\mathcal{P}$ counterclockwise by $e_1, \dots, e_k$ and for each $j$, $1 \le j \le k$, let $v_j := (v_{j1}, v_{j2})$ be the smallest integral vector on the *outward pointing* normal to $e_j$ and let $\tilde{C}_j$ be the torus invariant curve associated to $e_j$ on the toric surface $X_\mathcal{P}$ corresponding to $\mathcal{P}$. Let $\mathcal{I}$ be the $k \times k$ matrix of intersection numbers $(\tilde{C}_i, \tilde{C}_j)$ for $1 \le i, j \le k$. Each $v_j$ corresponds to a toric surface $\bar{X}_j$ corresponding to a triangle $\mathcal{P}_j$ that has two edges along the positive axes and the other edge is parallel to $e_j$. Then an application of Lemma 3.6 to $\bar{X}_1, \dots, \bar{X}_k$ and $\bar{X} := X_\mathcal{P}$ yields the following.

Corollary 3.7.

$$\mathcal{I} = \begin{pmatrix} 1 & \frac{v_{22}}{v_{12}} & \frac{v_{32}}{v_{12}} & \cdots & \frac{v_{k2}}{v_{12}} \\ \frac{v_{11}}{v_{21}} & 1 & \frac{v_{32}}{v_{22}} & \cdots & \frac{v_{k2}}{v_{22}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{v_{11}}{v_{k1}} & \frac{v_{21}}{v_{k1}} & \frac{v_{31}}{v_{k1}} & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{v_{11}v_{12}} & 0 & \cdots & 0 \\ 0 & \frac{1}{v_{21}v_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{v_{k1}v_{k2}} \end{pmatrix}.$$

Remark 3.8. We note that it is possible to compute the intersection numbers of the curves of infinity on *all* normal compactifications of $\mathbb{C}^2$ in terms of the *key polynomials* of the semidegrees corresponding to the curves at infinity. We refer to [Mon11b] for an explicit formula.

The final result of this section is Proposition 3.15 below. For this result (and also for Remark 3.8) we set $\mathbb{K} := \mathbb{C}$, because our proof uses the description of the valuations on $\mathbb{C}(x, y)$ in terms of associated Puiseux series (as developed *e.g.*, in [FJ04, Chapter 4]). It is however plausible that similar statements can be proved for fields with positive characteristic using Hamburger–Noether expansions.

Let $(x, y)$ be a fixed set of coordinates on $(\mathbb{C}^*)^2$ and $f$ be a Laurent polynomial in $(x, y)$ such that the origin is in the interior of the Newton polygon $\mathrm{NP}_{(x,y)}(f)$ of $f$ with respect to $(x, y)$-coordinates. Recall from Remark-Example 2.5 that $\mathcal{P} := \mathrm{NP}_{(x,y)}(f)$ determines a *rational subdegree* $\delta_{\mathcal{P}}$. The semidegrees associated to $\delta_{\mathcal{P}}$ are precisely the weighted degrees $\delta_E$ on $\mathbb{C}(x, y)$ determined by edges $E$ of $\mathcal{P}$. More precisely, if $E$ is an edge of $\delta_{\mathcal{P}}$ and $(a, b)$ is the *smallest outward pointing* normal to $E$ with *integral* coordinates, then $\delta_E$ assigns weights $a$ to $x$ and $b$ to $y$. We say that $\mathcal{E} := \{\delta_E : E \text{ is an edge of } \mathcal{P}\}$ is a *fundamental collection of semidegrees for $f$ in $(x, y)$ coordinates*. The BKK bound for the number of solutions of generic polynomials with Newton polygon $\mathcal{P}$ is a natural consequence of the intersection theory of the toric variety $X_{\mathcal{P}}$ determined by $\mathcal{E}$. Our final result extends the BKK non-degeneracy condition $(*)$ to the case of compactifications of $\mathbb{C}^2$ determined by maxima of weighted degrees in *different* sets of coordinates. In order to do it, at first we find an extension of the notion of a fundamental collection of semidegrees which applies to the latter scenario. Note that below when we say that a degree-like function $\delta$ is *normalized*, we mean that $\gcd\big(\delta(f) : f \in \mathbb{C}[x, y]\big) = 1$.

Definition 3.9. Let $\delta$ be a weighted degree on $\mathbb{C}(x, y)$ in $(x, y)$-coordinates corresponding to integral weights $a$ for $x$ and $b$ for $y$. For a given semidegree $\eta$ on $\mathbb{C}[x, y]$ and $c := (c_1, c_2) \in (\mathbb{C}^*)^2$, we say that $\delta$ *approximates $\eta$ by $c$* if the following conditions hold:

(i) There is $r \in \mathbb{Q}$, $r > 0$, such that $\eta(x^\alpha y^\beta) = r\delta(x^\alpha y^\beta)$ for all $(\alpha, \beta) \in \mathbb{Z}^2$.

(ii) Let $p, q$ be relatively prime integers such that $d := pa = qb$. Define $f := c_2^q x^p - c_1^p y^q$ (*i.e.*, $f$ is a Laurent polynomial of weighted degree $d$ such that $f(c) = 0$). Then $\eta(f) < dr = r\delta(f)$.

REMARK 3.10. Let $\delta$ and $\eta$ be as in Definition 3.9. Note that $-\eta$ and $-\delta$ are *discrete valuations* on $\mathbb{C}(x,y)$. In the notation of [FJ04], $\delta$ approximates $\eta$ by $c$ iff $f/c_2^q$ is precisely the first *key polynomial* of $-\eta$ in $(x,y)$ coordinates.

DEFINITION 3.11. Fix a finite collection $\mathcal{C}$ of polynomial coordinates on $X := \mathbb{C}^2$ and $f \in \mathbb{C}[X]$. Then a *fundamental collection of semidegrees for $f$ with respect to $\mathcal{C}$* is a collection $\mathcal{E}$ of normalized weighted degrees in systems of coordinates in $\mathcal{C}$ such that

(i) every semidegree $\delta$ in $\mathcal{E}$ is *centered at infinity*, *i.e.*, $\delta(f) > 0$ for some $f \in \mathbb{C}[x,y]$, and
(ii) for each system of coordinates $(x,y) \in \mathcal{C}$ and for each edge $E$ of $\mathrm{NP}_{x,y}(f)$ such that the outward pointing normal to $E$ has at least one positive coordinate, at least one of the following conditions holds:
   (a) $\delta_E \in \mathcal{E}$, or
   (b) for each zero $c \in (\mathbb{C}^*)^2$ of the leading weighted homogeneous form of $f$ with respect to $\delta_E$, there is $\eta \in \mathcal{E}$ such that $\delta_E$ approximates $\eta$ by $c$.

REMARK 3.12. Note that fundamental collections of weighted degrees with respect to a given $\mathcal{C}$ are *not* unique, since adding a new weighted degree to a fundamental collection creates another fundamental collection. However, for a given $\mathcal{C}$ and $f$, there is always a unique *minimal* fundamental collection; in the case that $\mathcal{C}$ consists of only one coordinate system $(x,y)$, the minimal fundamental collection is precisely the collection of all $\delta_E$'s corresponding to edges $E$ of $\mathrm{NP}_{x,y}(f)$ such that the outward pointing normal to $E$ has at least one positive coordinate. Note that we only consider the edges $E$ for which the outward pointing normals have at least one positive coordinate, since we are concerned here with $\mathbb{C}^2$ (instead of $(\mathbb{C}^*)^2$ as in the setting of Bernstein's theorem) and it is precisely for these edges $E$ that the corresponding semidegrees $\delta_E$ are centered at infinity (*i.e.*, they correspond to orders of pole along curves at infinity on a compactification of $\mathbb{C}^2$).

EXAMPLE 3.13. Let $f := (x^3-y^2-y)(x-y^2-y)-1 \in \mathbb{C}[x,y]$. See Figure 1 for the Newton polygons of $f$ in respectively $(x,y)$ and $(x',y')$-coordinates, where $x' := x - y^2$ and $y' := y$ (note that in the notation of Figure 1, $f$ corresponds to $p_k = q_k = 1$). As implied by Remark 3.12, the minimal fundamental collection of semidegrees for $f$ with respect to $(x,y)$ (resp. $(x',y')$) coordinates is $\{\delta_{E_1}, \delta_{E_2}\}$ (resp. $\{\delta_{E'_1}, \delta_{E'_2}\}$). On the other hand, the minimal fundamental collection of semidegrees for $f$ with respect to $\mathcal{C} := \{(x,y),(x',y')\}$ is $\{\delta_{E_1}, \delta_{E'_2}\}$.

REMARK 3.14 (A Geometric Interpretation). If $\bar{X}$ is a normal compactification of $X := \mathbb{C}^2$ and $C$ is a curve at infinity on $\bar{X}$ such that its associated semidegree (*i.e.*, the order of pole along $C$) is a weighted degree in a system of coordinates $(x,y)$ on $\mathbb{C}^2$, then $C$ has two *marked* points $P_0$ and $P_\infty$: if $\mathcal{L}$ is the *pencil of curvettes* which intersect $C$ at generic points, then $P_0$ (resp. $P_\infty$) is the point on $C$ corresponding to the *zeroth* curvette (resp. the *curvette at infinity*) [Mon, Section 2]. Assume that each semidegree corresponding to a curve at infinity on $\bar{X}$ is a weighted degree in a system of coordinates in $\mathcal{C}$ and that for at least one system of coordinates $(x, y)$ in $\mathcal{C}$, neither $x$ nor $y$ divides $f$ in $\mathbb{C}[x, y]$. Then the semidegrees associated with curves at infinity on $\bar{X}$ constitute a fundamental collection for $f$ with respect to $\mathcal{C}$ if and only if *no* point at infinity on the closure $\bar{C}_f$ in $\bar{X}$ of the curve $C_f$ defined by $f$ (on $X$) is a marked point on some curve at infinity. In particular, $\bar{X}$ is non-singular at every point of $\bar{C}_f$.

The following result of [Mon11a] generalizes the BKK non-degeneracy criterion to subdegrees defined by weighted degrees in possibly distinct coordinates.

PROPOSITION 3.15. *Let $X := \mathbb{C}^2$, $f_1, f_2 \in \mathbb{C}[X]$, and $\mathcal{C}$ be a collection of polynomial coordinates on $X$. For each $k$, $1 \leq k \leq 2$, let $\mathcal{E}_k$ be a fundamental collection of semidegrees for $f_k$ with respect to $\mathcal{C}$ and $L_k := \{f \in \mathbb{C}[X] : \delta(f) \leq \delta(f_k)$ for all $\delta \in \mathcal{E}_k\}$. Assume that for each $k$, $1 \leq k \leq 2$, and every system of coordinates $(x, y)$ in $\mathcal{C}$,*
(i) *the Newton polygon of $f_k$ with respect to $(x, y)$ is full dimensional, and*
(ii) *neither $x$ nor $y$ divides $f$ in $\mathbb{C}[x, y]$.*
*Then $f_1, f_2$ are non-degenerate with respect to $L_1, L_2$ (in the sense of Step 1 of the introduction) iff the following condition holds:*

(**) *For each $\delta \in \mathcal{E}_1 \cup \mathcal{E}_2$ and each common zero $c \in (\mathbb{C}^*)^2$ of the leading forms of $f_1$ and $f_2$ with respect to $\delta$, there is $\eta \in \mathcal{E}_1 \cup \mathcal{E}_2$ such that $\delta$ approximates $\eta$ by $c$.*

REMARK 3.16. It follows from the arguments of the following proof that $L_1$ and $L_2$ of Proposition 3.15 are finite-dimensional vector spaces over $\mathbb{C}$.

SKETCH OF A PROOF OF PROPOSITION 3.15. Fix $k$, $1 \leq k \leq 2$. We claim that there is a (unique) normal projective compactification $\bar{X}_k$ of $X$ such that $\mathcal{E}_k$ is precisely the collection of normalized semidegrees associated to irreducible curves at infinity on $\bar{X}_k$. Indeed, pick any non-singular compactification $\bar{X}_{k,0}$ of $X$ such that each semidegree $\delta$ in $\mathcal{E}_k$ corresponds to a curve $C_\delta$ at infinity on $\bar{X}$. We have to show that it is possible to (projectively) contract the 'extra' curves at infinity on $\bar{X}_{k,0}$, *i.e.*, those irreducible components of $X_{k,0} \setminus X$ that do not belong to $S_k := \{C_\delta : \delta \in \mathcal{E}_k\}$. Indeed, it can be shown (using Remark 3.14 and properties of compactifications of $\mathbb{C}^2$) that every 'extra' curve $C$ at infinity can be contracted if in addition $C$ intersects the closure $\bar{C}_k$ in $\bar{X}_{k,0}$ of the curve $C_k$ on $X$ defined by $f_k$. Therefore we may assume that $\bar{C}_k$ does not intersect any 'extra' curve at infinity. Let $D_k$ be the divisor of poles of $f_k$ on $\bar{X}_{k,0}$. Since the base locus of $D_k$ is finite, it follows from the Zariski–Fujita theorem that $mD_k$ is base-point free for some $m \geq 1$. It can be shown using hypothesis (i) on $f_k$ that for some $l \geq 1$, the global sections of $\mathcal{O}_{\bar{X}_{k,0}}(lmD_k)$ contains elements which are algebraically independent over $\mathbb{C}(f_k)$. Let $\phi\colon \bar{X}_{k,0} \to \mathbb{P}^N$ be the morphism determined by global sections of $\mathcal{O}_{\bar{X}_{k,0}}(lmD_k)$ and $\bar{X}_{k,0} \xrightarrow{\phi_1} \bar{X}_{k,1} \to \mathbb{P}^N$ be the Stein factorization of $\phi$. Then $\phi_1$ contracts every 'extra' curve $C$ at infinity, since the divisor of zeroes of $f_k^{lm}$ (which is a global section of $\mathcal{O}_{\bar{X}_{k,0}}(lmD_k)$) does not intersect $C$. It is of course possible that $\phi_1$ also contracts some curves in $S_k$, but it follows from the theory of surfaces that the matrix of intersection numbers of 'extra' curves at infinity is negative definite, and therefore it is possible to contract *only* them, resulting in a normal analytic surface $\bar{X}_k$. It then can be shown, *e.g.*, using the results in [Mon11b], that $\bar{X}_k$ is projective as claimed in the beginning of the paragraph.

Getting back to the proof of the proposition, let $\bar{X}$ be the normalization of the closure of the diagonal image of $X$ into $\bar{X}_1 \times \bar{X}_2$. Then condition $(**)$ implies that $\bar{X}$ preserves the intersection of the curves $V(f_1), V(f_2) \subseteq X$ at infinity (this can be seen by an argument similar to those needed to prove Remark 3.14). The proposition now follows from applying Lemma 1.3 with setting $D_k$ to be the divisor of poles of $f_k$ on $\bar{X}_k$, $1 \leq k \leq 2$. □

The arguments of the proof of Proposition 3.15 yield the following.

COROLLARY 3.17. *Let $X$, $\mathcal{C}$ and $\mathcal{E}_k$, $1 \leq k \leq 2$, be as in Proposition* 3.15. *Then for each $k$, $1 \leq k \leq 2$, $\mathcal{E}_k$ is the collection of semidegrees corresponding to curves at infinity on a projective compactification $\bar{X}_k$ of $X$. Let $\bar{X}$ be the closure of the diagonal embedding of $X$ into $\bar{X}_1 \times \bar{X}_2$. Then for all $f_1, f_2 \in \mathbb{C}[X]$, the number of isolated solutions of $f_1, f_2$ on $X$ equals the intersection number of the divisors of poles of $f_j$ on $\bar{X}$ if and only if $(**)$ holds.* □

REMARK 3.18. In the situation of Proposition 3.15, if in addition every semidegree in $\mathcal{E}_1 \cup \mathcal{E}_2$ is itself *projective*, then it follows from Corollary 3.17 that the number of solutions of generic polynomials from $L_1$ and $L_2$ can be computed using Lemma 3.6.

## 4. Counting the number of solutions of a BKK degenerate polynomial system on $\mathbb{C}^2$: an example.

Consider polynomials $f_1 := (x^3 - y^2 - y)^{p_1}(x - y^2 - y)^{q_1} - a_1$ and $f_2 := (x^3 - 2y^2 - y)^{p_2}(x - y^2 - 2y)^{q_2} - a_2$ for some $p_1, p_2, q_1, q_2 \geq 1$ and $a_1, a_2 \in \mathbb{C}$. Then $f := (f_1, f_2)$ fails the BKK non-degeneracy criterion, since the leading weighted homogeneous forms of $f_k$'s corresponding to the weighted degree with weight vector $(2,1)$ for $(x,y)$ are of the form $x^{3p_k}(x - y^2)^{q_k}$ which have a common solution $(1,1) \in (\mathbb{C}^*)^2$ (see Figure 1 for the Newton polygon). Since the common zero arises as a solution of $x - y^2$, this suggests that we should look for a change of coordinates of the form $x' := x - y^2$, $y' := y$. It turns out that $(f_1, f_2)$ is BKK degenerate even in $(x', y')$-coordinates (the degeneracy coming from the edge $E_1'$ in Figure 1), but this change of coordinate does resolve the degeneracy along the direction of the 'initial problematic edge' $E_2$. This suggests that we should consider *both* coordinates $(x, y)$ and $(x', y')$.

Indeed, it is straightforward to check that $\mathcal{E} := \{\delta, \delta'\}$ is a fundamental collection for both $f_1$ and $f_2$ with respect to $\{(x,y), (x',y')\}$, and that the non-degeneracy condition $(**)$ is satisfied with $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. We now compute the number of zeros of $f_1 - a_1$ and $f_2 - a_2$ using Corollary 3.17. Since both $\mathcal{E}_1$ and $\mathcal{E}_2$ equal $\mathcal{E}$, it follows that the surfaces $\bar{X}_1, \bar{X}_2$ and $\bar{X}$ of Corollary 3.17 are

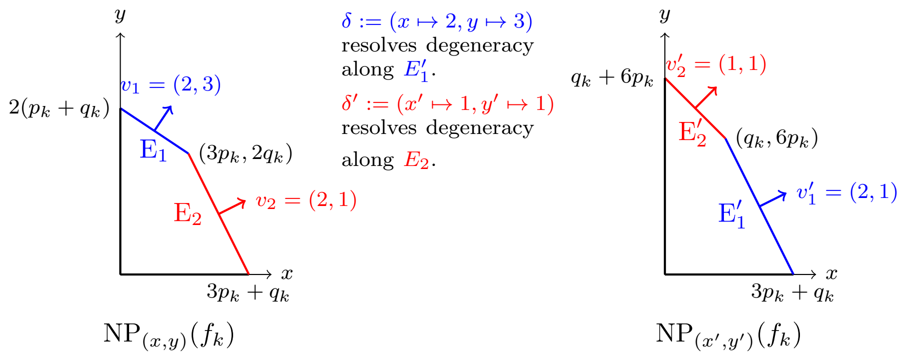



*Figure 1*: Newton polygons of $f_k$ in $(x,y)$ and $(x',y')$ coordinates.

isomorphic. Moreover, in this case both semidegrees $\delta$ and $\delta'$ are *projective*: the compactification $\bar{X}^{\delta}$ of $\mathbb{C}^2$ corresponding to $\delta$ is the weighted projective space $\mathbb{P}^2(1,2,3)$ and the compactification $\bar{X}^{\delta'}$ corresponding to $\delta'$ is simply $\mathbb{P}^2$. Therefore, as stated in Remark 3.18, we can apply Lemma 3.6 to compute intersection products of the curves at infinity on $\bar{X}$. Let $C$ (resp. $C'$) be the complement of $\mathbb{C}^2$ in $\bar{X}^{\delta}$ (resp. $\bar{X}^{\delta'}$). Then $(C,C)=\frac{1}{6}$ and $(C',C')=1$. Moreover,

$$\begin{aligned} l_\infty(\delta,\delta') &:= \max\Big\{\frac{\delta'(f)}{\delta(f)} : f \in \mathbb{C}[x,y],\ \delta(f)>0\Big\} \\ &= \max\Big\{\frac{\delta'(x)}{\delta(x)}, \frac{\delta'(y)}{\delta(y)}\Big\} = \max\Big\{\frac{2}{2}, \frac{1}{3}\Big\} = 1, \\ l_\infty(\delta',\delta) &:= \max\Big\{\frac{\delta(f)}{\delta'(f)} : f \in \mathbb{C}[x,y],\ \delta'(f)>0\Big\} \\ &= \max\Big\{\frac{\delta(x')}{\delta'(x')}, \frac{\delta(y')}{\delta'(y')}\Big\} = \max\Big\{\frac{6}{1}, \frac{3}{1}\Big\} = 6. \end{aligned}$$

Denote the strict transform of $C$ (resp. $C'$) on $\bar{X}$ by $C_1$ (resp. $C_2$). Then Lemma 3.6 implies that the matrix of intersection products $(C_i,C_j)$ is

$$\mathcal{I} = \begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{30} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}.$$

For each $k$, the polar divisor of $f_k - a_k$ (with $a_k \in \mathbb{C}$) on $\bar{X}$ is $D_k := 6(p_k + q_k)[C_1] + (6p_k+q_k)[C_2]$. Consequently, Corollary 3.17 implies that for all $a_1, a_2 \in \mathbb{C}$ the number of solutions in $\mathbb{C}^2$ of $f_1 - a_1$ and $f_2 - a_2$ is precisely $(D_1, D_2) = 6p_1p_2 + 6p_1q_2 + 6q_1p_2 + q_1q_2$. We state for the sake of completeness that the mixed volumes of the Newton polygons of $f_1$ and $f_2$ in $(x,y)$ and $(x',y')$ coordinates are respectively $\frac{1}{2}(6p_1p_2+6p_1q_2+6q_1p_2+2q_1q_2)$ and $\frac{1}{2}(18p_1p_2+6p_1q_2+6q_1p_2+q_1q_2)$, so that the BKK bounds in both coordinates are greater than the actual number of solutions, as expected.

## References


[**Ber75**] D. N. Bernstein, *The number of roots of a system of equations.* Funkcional. Anal. i Priložen **9** (1975), 1–4.

[**Dam99**] James Damon, *A global weighted version of Bezout's theorem.* In: The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun. **24**, Amer. Math. Soc., Providence, RI, 1999, 115–129.

[**FJ04**] Charles Favre and Mattias Jonsson, *The valuative tree.* Lecture Notes in Math. **1853**, Springer-Verlag, Berlin, 2004.

[**Ful98**] William Fulton, *Intersection theory.* Second edition, Ergeb. Math. Grenzgeb. **2**, Springer-Verlag, Berlin, 1998.

[**HS95**] Birkett Huber and Bernd Sturmfels, *A polyhedral method for solving sparse polynomial systems.* Math. Comp. **64** (1995), 1541–1555.

[**HS97**] ———, *Bernstein's theorem in affine space.* Discrete Comput. Geom. **17** (1997), 137–141.

[**Huc70**] James A. Huckaba, *Some results on pseudo valuations.* Duke Math. J. **37** (1970), 1–9.

[**Kho77**] A. G. Khovanskii, *Newton polyhedra and toroidal varieties.* Funkcional. Anal. i Priložen. **11** (1977), 56–64, 96.

[**Kho78**] ———, *Newton polyhedra, and the genus of complete intersections.* Funktsional. Anal. i Prilozhen. **12** (1978), 51–61.

[**KK08**] Kiumars Kaveh and A. G. Khovanskii, *Mixed volume and an analogue of intersection theory of divisors for non-complete varieties.* http://[math.AG], 2008.

[**Kus76**] A. G. Kushnirenko, *Newton polyhedra and Bezout's theorem.* Funkcional. Anal. i Priložen. **10** (1976), 82–83.

[**Laz04**] Robert Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series.* Ergeb. Math. Grenzgeb. **48**, Springer-Verlag, Berlin, 2004.

[**LW96**] T. Y. Li and Xiaoshen Wang, *The BKK root count in* $\mathbf{C}^n$. Math. Comp. **65** (1996), 1477–1484.

[**Mon**] Pinaki Mondal, *Normal compactifications of the affine plane via pencils of jets of curves.* C. R. Math. Acad. Sci. Soc. R. Can., to appear.

[**Mon10a**] ———, *Projective completions of affine varieties determined via 'degree-like' functions.* arXiv:1012.0835.

[**Mon10b**] ———, *Towards a Bezout-type theory of affine varieties.* Ph.D. thesis, University of Toronto, 2010.

[**Mon11a**] ———, *General bezout-type theorems.* arXiv:1111.0614.

[**Mon11b**] ———, *Primitive normal compactifications of the affine plane I.* arXiv:1110.6905.

[**Mum61**] David Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.

[**Roj94**] J. Maurice Rojas, *A convex geometric approach to counting the roots of a polynomial system.* In: Selected papers of the Workshop on Continuous Algorithms and Complexity (Barcelona, 1993), Theoret. Comput. Sci. **133** (1994), 105–140.

[**Roj99**] ———, *Toric intersection theory for affine root counting.* J. Pure Appl. Algebra **136** (1999), 67–100.

[**RW96**] J. Maurice Rojas and Xiaoshen Wang, *Counting affine roots of polynomial systems via pointed Newton polytopes.* J. Complexity **12** (1996), 116–133.

[**Sam59**] Pierre Samuel, *Multiplicités de certaines composantes singulières.* Illinois J. Math. **3** (1959), 319–327.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