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Systems of Linear Partial Differential Equations with Constant Coefficients: Bounds on Solutions

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Abstract. We consider sequences of systems of first-order linear partial differential equation with constant coefficients. As the index N of the sequence increases, the dimension of the integration space of the N^{th} system increases to infinity. Even though the coefficient matrices of the systems have different dimensions, the matrices originate from a common generating process that is specified by a finite sequence of real numbers or a polynomial. We establish results on the limiting behaviour of the L^2 norms of solutions of the systems as N grows to infinity, specifically we find asymptotic dominating powers of N for the L^2 norms. We show that the exponents of N depend on the maximal order of the complex unit roots of the generating polynomial. Our results are generalizations of some inequalities shown by Varadhan in his work on the hydrodynamic scaling limit of a non-gradient particle system with continuous spin and nearest-neighbour interaction. Our results could prove relevant for the derivation of hydrodynamic limits of other particle systems with complex interactions and non-Gaussian equilibrium measures.

Résumé. Nous considérons des séquences de systèmes d'équations aux dérivées partielles linéaires du premier ordre à coefficients constants. Lorsque l'indice N de la séquence s'accroît, la dimension de l'espace d'intégration du N-ième système s'approche de l'infini. Même si les matrices de coefficients des systèmes ont des dimensions différentes, les matrices proviennent d'un processus de génération qui est spécifié par une séquence finie de nombres réels ou un polynôme. Nous montrons des résultats sur le comportement asymptotique pour les normes L^2 des solutions des systèmes quand N s'approche de l'infini, spécifiquement nous trouvons des puissances de N qui dominent asymptotiquement les normes L^2. Nous montrons que les exposants de N dépendent de l'ordre maximal des racines complexes unités du polynôme génératrice. Nos résultats sont des généralisations des certaines inégalités montrées par Varadhan dans son travail sur la limite hydrodynamique d'une système de particules avec spin continu et évoluant selon une dynamique de Kawasaki non-gradient. Nos résultats pourraient se révéler pertinents pour le calcul des limites hydrodynamiques d'autres systèmes de particules avec des interactions complexes et des mesures d'équilibre non-gaussiennes.

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1. Introduction To derive the hydrodynamic scaling limit, i.e., the behaviour on the macroscopic scales, of a non-gradient random interacting particle system, the non-vanishing part of microscopic fluctuations at the macroscopic scales must be identified. Complex mathematical methods are proposed and successfully applied to rigorously explain the physical result, in works of Varadhan [10] for continuous spin systems, Quastel [5], [7] and Quastel and Yau [6], for discrete spin system. In the case of a particle system with continuous spin and nearest neighbour interactions, Varadhan [10] uses results on a sequence of systems of first-order linear partial differential equations with constant coefficients, specifically bounds on the limiting behaviour of the L^2-norm of the system solutions as the dimension of the integration space grows to infinity.

In an attempt to generalize Varadhan's result to more complex particle systems, not restricted to nearest neighbour dynamics, we establish bounds on L^2-norms of solutions of more general sequences of systems of first-order linear partial differential equations. The results, to be explained in the present paper, would be relevant, especially, for the continuum solid-on-solid model, [8], with non-Gaussian equilibrium measure, a model for the phenomenon of surface diffusion observed in molecular beam epitaxy, [2]. The hydrodynamic scaling limit and the decomposition of microscopic fluctuations of solid-on-solid model with Gaussian equilibrium measure were previously derived by Savu in [8] and [9], using Fourier analysis techniques.

Next we describe the sequence of systems considered by Varadhan in [10], and several generalized system sequences to be discussed in the present paper.

We consider a probability measure $\mu(dx) = \exp(-\phi(x))dx$, not necessarily Gaussian, on the space of real numbers, $\mathbb{R}$, and the product probability measure

$$\begin{aligned} \mu^N(x_1,\ldots,x_N) &= \exp\left(-\phi_n(x_1,\ldots,x_N)\right) dx_1\ldots dx_N \\ &= \exp\left(-\sum_{i=1}^N \phi(x_i)\right) dx_1\ldots dx_N \end{aligned}$$

on $\mathbb{R}^N$. The density-defining function $\phi:\mathbb{R}\to\mathbb{R}$ is required to have continuous first and second derivatives and the property that there exists some constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \le \phi^{''}(x) \le C_2 \quad \text{for all} \quad x\in\mathbb{R}. \tag{1}$$

The system sequence considered by Varadhan in [10] is indexed by integers $N \ge 2$ and has N^{th} term

$$\left\{\begin{array}{lcl} \partial_{x_1} g - \partial_{x_2} g &=& \xi_1^N \\ \partial_{x_2} g - \partial_{x_3} g &=& \xi_2^N \\ \quad\ldots && \\ \partial_{x_{N-1}} g - \partial_{x_N} g &=& \xi_{N-1}^N \end{array}\right. \tag{2}$$

where $\xi_1^N,\ldots,\xi_{N-1}^N$ are functions from $L^2(\mathbb{R}^N,\mu^N)$ and $\partial_{x_i} g$ is the weak partial derivative of g with respect to the coordinate x_i for any $1\le i\le N$. The solution

g is searched in the Sobolev space $W^{1,2}(\mathbb{R}^N, \mu^N)$ of functions with first-order weak partial derivatives that are square integrable. Square integrability above is defined with respect to μ^N. In order for the system (2) to admit at least one solution, ξ's must satisfy

$$\partial_{x_j} \xi_i^N - \partial_{x_{j+1}} \xi_i^N = \partial_{x_i} \xi_j^N - \partial_{x_{i+1}} \xi_j^N \tag{3}$$

in the weak sense, for any $1 \leq i, j \leq N-1$. Varadhan considered a certain infinite array of ξ's, to be described later, and found for the L^2-norm of specific solutions, asymptotic dominating powers of N, N being the dimension of the integration space.

In this paper we discuss systems like

$$A \nabla g = \xi \tag{4}$$

where the coefficient matrix $A = (a_{ij} : 1 \leq i \leq m, 1 \leq j \leq N)$ is a $m \times N$-matrix, $m \leq N$, with constant entries and independent rows, and ξ is a vector of m functions from $L^2(\mathbb{R}^N, \mu^N)$. The solution g is searched in the Sobolev space $W^{1,2}(\mathbb{R}^N, \mu^N)$, as the partial differential equations are understood in the weak sense in $L^2(\mathbb{R}^N, \mu^N)$. Because of commutativity of partial derivatives a necessary condition for (4) to admit at least one solution is

$$a_{j1} \frac{\partial \xi_i}{\partial x_1} + a_{j2} \frac{\partial \xi_i}{\partial x_2} + \cdots + a_{jN} \frac{\partial \xi_i}{\partial x_N} = a_{i1} \frac{\partial \xi_j}{\partial x_1} + a_{i2} \frac{\partial \xi_j}{\partial x_2} + \cdots + a_{iN} \frac{\partial \xi_j}{\partial x_N} \tag{5}$$

for all $1 \leq i, j \leq m$. In the remaining of the paper we implicitly assume that ξ's satisfy (5). The general solution of the system (4) is described in section 2 and an inequality that provides upper bound for the L^2-norm of a specific solution g_0 is obtained in section 3. The inequality is reminiscent of Poincaré inequality for probability measures on $\mathbb{R}^N$, [1], and can be considered a generalization of this classic inequality.

In section 3 we restrict the matrix A to be sparse and consider sequences of sparse matrices A and vectors ξ. As the sequence index N increases the dimensions of A and ξ grow to infinity. However, all matrices A and vectors ξ in the sequence share a common generating process, as explained next. Let $p_0, \ldots, p_r$ be a sequence of real numbers with $p_0 \neq 0$ and $p_r \neq 0$ and N be a large natural number such that $N \geq r+1$. We use the sequence p to construct

$$T_N(p) = \begin{pmatrix} p_0 & p_1 & \ldots & p_r & & & & \\ & p_0 & p_1 & \ldots & p_r & & & \\ & & \ldots & \ldots & \ldots & \ldots & & \\ & & & p_0 & p_1 & \ldots & p_r \end{pmatrix} \tag{6}$$

a $(N-r) \times N$ sparse, upper-triangular matrix whose non-zero entries are confined to the diagonals $0, 1, \ldots, r$. The system (4) with $A = T_N(p)$ is a natural generalization of Varadhan's system (2). The necessary condition (5) for (4) to admit at least a solution becomes

$$(7)\quad p_0\frac{\partial \xi_i}{\partial x_j}+p_1\frac{\partial \xi_i}{\partial x_{j+1}}+\cdots+p_r\frac{\partial \xi_i}{\partial x_{j+r}}=p_0\frac{\partial \xi_j}{\partial x_i}+p_1\frac{\partial \xi_j}{\partial x_{i+1}}+\cdots+p_r\frac{\partial \xi_j}{\partial x_{i+r}}$$

for all $1 \le i, j \le N-r$, if $A = T_N(p)$. For the sequence of systems (4) with $A = T_N(p)$, we characterize the limiting behaviour of the L^2-norms of specific solutions g_0 as the dimension N approaches infinity, namely, we found asymptotic dominating powers of N for the L^2-norms of g_0.

2. Systems with General Coefficient Matrices

2.1. The Solutions. Systems (4), of first-order linear partial differential equations with constant coefficients have been around for a long time and can be considered one of the simplest type of systems of partial differential equations. Their general solutions can be easily characterized by a geometric argument.

The linearity of system (4) implies that its general solution can be simply described as the sum of a particular solution and the general solution of the homogeneous system

$$(8)\qquad A\nabla g = 0.$$

Reducing (4) to solving a homogeneous system has the advantage that a geometric point of view can be adopted. The homogeneous system, (8), says that the directional derivatives of g in the m directions specified by the m rows of A are 0. Thus, g must be constant in any of these m directions, or equivalently g must be constant on any m-dimensional flat of $\mathbb{R}^N$ parallel to the m directions. Let V be the vector subspace of $\mathbb{R}^N$ generated by the m rows of A. Let B be a $(N-m)\times N$-matrix whose rows $b_1,\ldots,b_{N-m}$ are independent vectors in $\mathbb{R}^N$ that are orthogonal on any vector of V. Any m-dimensional flat of $\mathbb{R}^N$ parallel to V is of the form $x+V$ for some x in $\mathbb{R}^N$ and can be uniquely described by $N-m$ real numbers $\beta_1,\ldots,\beta_{N-m}$ as the set

$$(9)\qquad \beta_1 b_1+\cdots+\beta_{N-m}b_{N-m}+V.$$

Given a point $x\in\mathbb{R}^N$, there exists a unique m-dimensional flat, (9), that passes through the point x and this unique flat is uniquely characterized by the values of the inner products: $x\cdot b_1,\ldots,x\cdot b_{N-m}$. Thus the general solution of the homogeneous system (8) can be described by

$$(10)\qquad g(x) = h(x\cdot b_1,\ldots,x\cdot b_{N-m})$$

for some h, a function in $W^{1,2}(\mathbb{R}^{N-m},\mu^{N-m})$.

Let g be a solution of the inhomogeneous system (4). We define a new function g_1 that is constant on any of the m-dimensional flats mentioned before; the value of g_1 on some flat being the average value of g on the flat. Specifically, if $x = A^T\gamma + B^T\beta \in \mathbb{R}^N$, for some $\gamma \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^{N-m}$ then

$$g_1(x) = g_1(A^T\gamma + B^T\beta) = \frac{\int_{\mathbb{R}^m} g(A^T\alpha + B^T\beta)e^{-\phi_N(A^T\alpha+B^T\beta)}d\alpha}{\int_{\mathbb{R}^m} e^{-\phi_N(A^T\alpha+B^T\beta)}d\alpha}. \tag{11}$$

Now $g_0 = g - g_1$ is another solution of the inhomogeneous system (4) that has the property that its average values along the m-dimensional flats are zero, i.e.

$$\int_{\mathbb{R}^m} g_0(A^T\alpha + B^T\beta)e^{-\phi_N(A^T\alpha+B^T\beta)}d\alpha = 0. \tag{12}$$

for any $\beta \in \mathbb{R}^{N-m}$. This property characterizes uniquely g_0 among the infinite set of solutions of (4).

2.2. Upper bound on the L^2-norm of the special solution g_0. Upper bounds for the L^2-norm, $\|g_0\|_2$, of the special solution, g_0, of the general system (4) can be obtained.

THEOREM 1. *Let g_0 be the specific solution of the differential system (4), uniquely characterized by the property that its average values along any of the m-dimensional flats parallel to the rows of A equal 0. Let U be an invertible matrix of size m. Then the following inequality holds:*

$$\int_{\mathbb{R}^N} g_0^2 d\mu^N \leq \frac{C}{\lambda_{min}(UA(UA)^T)} \int_{\mathbb{R}^N} |UA\nabla g_0|^2 d\mu^N. \tag{13}$$

Above C is a constant dependent on ϕ and independent of g_0, A or U. $\lambda_{min}(UA(UA)^T)$ stands for the smallest eigenvalue of the matrix $UA(UA)^T$.

Proof. First, we prove the result if U is the identity matrix. We select a m-dimensional flat, $F(\beta)$, that is parallel to the rows of A. The flat can be described as the set of points

$$\mathrm{F}(\beta) = \left\{x \in \mathbb{R}^N \;\mid\; x = A^T\alpha + B^T\beta,\;\; \alpha \in \mathbb{R}^m\right\}, \tag{14}$$

for some fixed $\beta \in \mathbb{R}^{N-m}$. Above α can be thought of as the intrinsic coordinates of $\mathrm{F}(\beta)$.

Next we show that the restriction of the measure μ^N on $\mathrm{F}(\beta)$ satisfies the Bakry-Emery criterion, [1]. The restriction of μ^N has density of the form $\exp(-\psi(\alpha))$ with $\psi(\alpha) = \phi_N(A^T\alpha + B^T\beta)$. The Hessian of ψ relates to the Hessian of ϕ and satisfies the next inequality, in the sense of quadratic forms,

$$\begin{aligned}(\mathrm{Hess}\psi)(\alpha) &\\ = \;& A(\mathrm{Hess}\phi_N)(A^T\alpha + B^T\beta)A^T \geq C_1AA^T \geq C_1\lambda_{min}(AA^T)Id_m.\end{aligned} \tag{15}$$

Above Id_m stands for the identity matrix of size m. The inequality (15) follows because the Hessian of ϕ_N is a diagonal matrix with $\phi^{''}(x_1), \ldots, \phi^{''}(x_N)$ on the main diagonal and $\phi^{''}$ was assumed to be bounded from below by C_1, implying that Hess $\phi \geq C_1 Id_N$, in the sense of quadratic forms.

The inequality (15) shows that the Bakry-Emery criterion, [1], is satisfied by the restriction of the measure μ^N on the flat $\mathrm{F}(\beta)$ and consequently, the Poincare inequality

$$\int\limits_{\mathbb{R}^m} f^2(\alpha)e^{-\psi(\alpha)}du \leq \frac{1}{C_1\lambda_{min}(AA^T)} \int\limits_{\mathbb{R}^m} |\nabla f|^2(\alpha)e^{-\psi(\alpha)}du \tag{16}$$

holds for any function f with zero average on $\mathrm{F}(\beta)$. In particular, if we take $f(\alpha) = g_0(A^T\alpha + B^T\beta)$, and $(\nabla f)(\alpha) = A(\nabla g_0)(A^T\alpha + B^T\beta)$, the inequality (16) becomes

$$\begin{aligned} &\int\limits_{\mathbb{R}^m} g_0^2(A^T\alpha + B^T\beta)e^{-\phi(A^T\alpha + B^T\beta)}d\alpha \\ &\leq \frac{1}{C_1\lambda_{min}(AA^T)} \int\limits_{\mathbb{R}^m} |A(\nabla g_0)(A^T\alpha + B^T\beta)|^2 e^{-\phi(A^T\alpha+B^T\beta)}d\alpha. \end{aligned} \tag{17}$$

We can integrate with respect to β in both sides of the inequality (17), as (17) holds independently of β, and apply a change of variable $x = A^T\alpha + B^T\beta$ to obtain

$$\int\limits_{\mathbb{R}^N} g_0^2 d\mu^N \leq \frac{1}{C_1\lambda_{min}(AA^T)} \int\limits_{\mathbb{R}^N} |A\nabla g_0|^2 d\mu^N. \tag{18}$$

Note that (18) follows after the Jacobian associated with the change of variable $x = A^T\alpha + B^T\beta$ is simplified from both sides of the inequality. Defining $C = 1/C_1$ we establish Theorem 1 for $U = Id_m$.

Second, for an arbitrary invertible matrix U, the arguments outlined above can be repeated with no modification once we observe that any m-dimensional flat can also be described as

$$\mathrm{F}(\beta) = \left\{ x \in \mathbb{R}^N \ \mid \ x = (UA)^T\alpha + B^T\beta, \ \ \alpha \in \mathbb{R}^m \right\}, \tag{19}$$

with β fixed in $\mathbb{R}^{N-m}$. Indeed $(UA)^T\alpha$ is still a linear combination of the rows of A. □

3. Systems with Sparse Coefficient Matrices In this section the coefficient matrix, A, of (4) is selected to be the sparse matrix $T_N(p)$, (6). In this case, the inequality (13) leads to asymptotic dominating powers of N, the dimension of the integration space, for the L^2-norm, $\|g_0\|_2$, of the specific solution

g_0. Mainly, this is a consequence of $T_N(p)$ being not only sparse but of a special structure such that $T_N(p)T_N(p)^T$ is a symmetric Toeplitz band matrix, [3]. Properties of eigenvalues of Toeplitz matrices are available, for example in [3]. Next we summarize some of the spectral properties of Toeplitz band matrices, that are relevant for our subsequent developments.

The symmetric Toeplitz band matrix, $T_N(p)T_N(p)^T$, of size $N-r$ has all its non-zero entries confined on the diagonals $-p,\dots,-1,0,1,\dots,p$ and is completely determined by the sequence $t_p,\dots,t_1,t_0,t_1,\dots,t_p$ where $t_j = p_0p_j + p_1p_{j+1} + \dots p_{r-j}p_r$, $j=0,\dots,r$. A Laurent polynomial can be constructed from the matrix defining sequence, namely

$$t(z) = t_r z^{-r} + \cdots + t_1 z^{-1} + t_0 + t_1 z + \cdots + t_r z^r \tag{20}$$

for any z on the complex unit circle $\mathbb{T}$. If we let $p(z)$ to be the polynomial $p_0 + p_1 z + \cdots + p_r z^r$, then the Laurent polynomial of $T_N(p)T_N(p)^T$ factors as $t(z) = p(z^{-1})p(z)$.

The roots in $\mathbb{T}$ of the Laurent polynomial and spectral properties of a symmetric Toeplitz band matrix are related. In the case of our Toeplitz matrix, $T_N(p)T_N(p)^T$, the coefficients of $p(z)$ are real numbers, and hence the roots in $\mathbb{T}$ of $t(z)$ and $p(z)$ coincides. Moreover, the order of any $\mathbb{T}$-root of $t(z)$ is twice the order of the same root of $p(z)$. Let α be the maximal order of the zeros of $p(z)$ in $\mathbb{T}$. If $p(z)$ has no root in $\mathbb{T}$ then α is taken to be 0. Based on these considerations the eigenvalue result from [3], page 245, can be stated in our context as

Proposition 1. *Let $p(z)$ be a real-coefficient polynomial. Then the lowest eigenvalue of $T_N(p)T_N(p)^T$ satisfies*

$$\lambda_{min}(T_N(p)T_N(p)^T) \simeq \frac{1}{N^{2\alpha}}, \tag{21}$$

where the notation $x_N \simeq y_N$ means that there are constants $D_1, D_2 \in (0,\infty)$ such that $D_1 y_N \le x_N \le D_2 y_N$ for all sufficiently large N. The constants D_1, D_2 depend on the coefficients of $p(z)$ and not on N.

We assume that for each $N \ge r+1$, we have a vector $\xi^N = (\xi_1^N,\dots,\xi_{N-r}^N)$ of $L^2(\mathbb{R}^N,\mu^N)$-functions that satisfies the necessary condition (7). Let g_0 be the specific solution of $T_N(p)\nabla g = \xi^N$ uniquely characterized by the property that it has zero average value along any $(N-r)$-dimensional flat of $\mathbb{R}^N$ that is parallel to the rows of $T_N(p)$. Obviously, g_0 depends on N, however, for the sake of notation simplicity, we omit N from the notation of g_0. We can combine Theorem 1 and Proposition 1 to conclude the following bound on $\|g_0\|_2$.

Corollary 1. *Assume that there exists a constant E such that $\|\xi_i^N\|_2 \le E$ for any $1 \le i \le N-r$ and $N \ge r+1$. Then there exists a constant C dependent on $p(z)$ and independent of g_0 and N, such that*

$$\int_{\mathbb{R}^N} g_0^2 d\mu^N \le C N^{2\alpha+1}. \tag{22}$$

Proof. From Theorem 1 with $U = Id_{N-r}$ and Proposition 1 , it follows

$$\begin{aligned} \int_{\mathbb{R}^N} g_0^2 d\mu^N &\leq \frac{C}{\lambda_{min}(T_N(p)T_N(p)^T)} \int_{\mathbb{R}^N} |T_N(p)\nabla g_0|^2 d\mu^N \\ &= \frac{C}{\lambda_{min}(T_N(p)T_N(p)^T)} \int_{\mathbb{R}^N} |\xi^N|^2 d\mu^N \\ &\leq \frac{C}{D_1} N^{2\alpha} \int_{\mathbb{R}^N} |\xi^N|^2 d\mu^N \leq \frac{CE}{D_1} N^{2\alpha+1}. \end{aligned} \tag{23}$$

Taking C to be CE/D_1 we establish the corollary. □

Varadhan in [10] selects a certain sequence of vectors ξ^N for the right sides of systems (2). Similarly we describe next a sequence of ξ's, in which case we are able to improve the bound of Corollary 1. Let $\mathbb{R}^{\mathbb{Z}}$ be the set of two-sided, real-valued sequences and τ be the shift that maps a two-sided sequence $x = (x_i : -\infty < i < \infty)$ into $\tau x = (x_{i+1} : -\infty < i < \infty)$. A shift operator on the space of functions defined on $\mathbb{R}^{\mathbb{Z}}$ is induced as follows: $(\tau\Xi)(x) = \Xi(\tau x)$ for any Ξ, a real-valued function with domain $\mathbb{R}^{\mathbb{Z}}$. Let τ^i be the i-fold composition of the shift operator. Let μ^∞ be the product measure of countable many copies of μ on $\mathbb{R}^{\mathbb{Z}}$ and Ξ be a function in $L^2(\mathbb{R}^{\mathbb{Z}}, \mu^\infty)$ with zero mean. In addition, we require Ξ to satisfy, in the weak sense the following relations

$$p_0 \frac{\partial \tau^i \Xi}{\partial x_j} + p_1 \frac{\partial \tau^i \Xi}{\partial x_{j+1}} + \cdots + p_r \frac{\partial \tau^i \Xi}{\partial x_{j+r}} = p_0 \frac{\partial \tau^j \Xi}{\partial x_i} + p_1 \frac{\partial \tau^j \Xi}{\partial x_{i+1}} + \cdots + p_r \frac{\partial \tau^j \Xi}{\partial x_{i+r}} \tag{24}$$

for all $-\infty \leq i, j \leq \infty$. Then for any $N \geq r+1$ and $-N \leq i \leq N-r$ we define

$$\xi_i^{2N+1} = E[\tau^i \Xi \,|\, x_{-N}, \ldots, x_N] \tag{25}$$

where by $E[\Xi \,|\, x_{-N}, \ldots, x_N]$ we mean the integral of Ξ with respect to the measure copies $\mu(dx_i)$, $|i| > N$. Note that each ξ_i^{2N+1}, $-N \leq i \leq N-r$ depends on $2N+1$ coordinates. Next g_0 stands for the specific solution of $T_{2N+1}(p)\nabla g = (\xi_{-N}^{2N+1}, \ldots, \xi_{N-r}^{2N+1})^T$ that has zero average value along any $(2N-r+1)$-dimensional flat parallel to the rows of $T_{2N+1}(p)$. The dependency on N of g_0, is not indicated in the solution notation.

Corollary 2. *Assume that p has only one $\mathbb{T}$-root, namely 1 with multiplicity $\alpha \geq 1$. Assume that the functions ξ_i^{2N+1}, $-N \leq i \leq N-r$, $N \geq r+1$ are selected as described in the previous paragraph. Then*

$$\lim_{N\to\infty} \frac{1}{N^{2\alpha+1}} \int_{\mathbb{R}^{2N+1}} g_0^2 d\mu^{2N+1} = 0. \tag{26}$$

Proof. To shorten the notations for the proof of this corollary, $T_N(p)$ stands for the band matrix (6) of dimension $(2N - r + 1) \times (2N + 1)$. From Theorem 1 with arbitrary invertible U we have the bound

$$\frac{1}{N^{2\alpha+1}} \int\limits_{\mathbb{R}^{2N+1}} g_0^2 d\mu^{2N+1} \leq \frac{C}{N^{2(\alpha-1)}\lambda_{min}(UT_N(p)(UT_N(p))^T)} \times \frac{1}{N} \int\limits_{\mathbb{R}^{2N+1}} \left|\frac{UT_N(p)\nabla g_0}{N}\right|^2 d\mu^{2N+1}. \tag{27}$$

Because of (27) the conclusion (26) of the corollary follows from two facts: (a) the existence of a constant C_3 independent of N such that

$$N^{2(\alpha-1)}\lambda_{min}(UT_N(p)(UT_N(p))^T) \geq C_3, \tag{28}$$

and (b) the validity of the limit

$$\lim_{N\to\infty} \frac{1}{N} \int\limits_{\mathbb{R}^{2N+1}} \left|\frac{UT_N(p)\nabla g_0}{N}\right|^2 d\mu^{2N+1} = 0. \tag{29}$$

Let U be the upper triangular matrix of size $(2N-r+1)\times(2N-r+1)$, that has all entries on and above the main diagonal equal to 1, namely

$$U = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{30}$$

The advantage of selecting this particular U is that, in comparison to $\lambda_{min}(T_N(p)T_N(p)^T)$, the lowest eigenvalue of $UT_N(p)(UT_N(p))^T$ is larger and decreases to 0 not faster than $1/N^{2(\alpha-1)}$. Precisely, we show next the existence of a constant C_3 independent of N such that (28) holds. To derive the inequality (28) we relate the lowest eigenvalues of $UT_N(p)(UT_N(p))^T$ and $T_N(q)T_N(q)^T$, where $q(z) = q_0 + \dots + q_{r-1}z^{r-1}$ is the polynomial defined by $p(z) = (z-1)q(z)$. As the degree of $p(z)$ is assumed to be r, we must have that $q_{r-1} \neq 0$.

Let $x = (x_i : -N \leq i \leq N - r)$ be an arbitrary point in $\mathbb{R}^{2N-r+1}$. If $r = 1$, matrix multiplication shows that the squared 2-norm of the $\mathbb{R}^{2N-r+1}$-point $(UT_N(p))^T x$ is

$$|(UT_N(p))^T x|^2 = |T_N(q)^T x|^2 + q_{r-1}^2\Big(\sum_{i=-N}^{N-r} x_i\Big)^2 \geq |T_N(q)^T x|^2. \tag{31}$$

If $r \geq 2$, matrix multiplications show that

$$|(UT_N(p))^T x|^2 = |(T_N(q) - B_N(q))^T x|^2 + q_{r-1}^2\Big(\sum_{i=-N}^{N-r} x_i\Big)^2, \tag{32}$$

with $B_N(q)$ being a $(2N-r+1)\times(2N)$-matrix.

$$B_{N_r}(q) = \begin{pmatrix} 0 & \dots & 0 & q_0 & \dots & q_{r-2} \\ 0 & \dots & 0 & q_0 & \dots & q_{r-2} \\ 0 & \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & q_0 & \dots & q_{r-2} \end{pmatrix}. \tag{33}$$

Further algebraic manipulations of (32) lead to

$$\begin{aligned} &|(UT_N(p))^T x|^2 \\ &\quad = |T_N(q)^T x|^2 - 2T_N(q)^T x \cdot B_N(q)^T x + |B_N(q)^T x|^2 + q_{r-1}^2 \Big(\sum_{i=-N}^{N-r} x_i\Big)^2 \\ &\quad = |T_N(q)^T x|^2 - 2T_N(q)^T x \cdot B_N(q)^T x + (q_0^2 + \dots + q_{r-1}^2) \times \Big(\sum_{i=-N}^{N-r} x_i\Big)^2. \end{aligned} \tag{34}$$

In the equality above all but the inner product $-2T_N(q)^T x \cdot B_N(q)^T x$ are positive terms. Next we use the common technique of completing a square to incorporate $-2T_N(q)^T x \cdot B_N(q)^T x$ into a positive term. Note that we can select $\tilde{q}_i^2 = q_i^2 + q_{r-1}^2/(r-1)$, $0 \le i \le r-2$, such that $\tilde{q}_0^2 + \dots + \tilde{q}_{r-2}^2 = q_0^2 + \dots + q_{r-1}^2$ and

$$|B_N(\tilde{q})^T x|^2 = (q_0^2 + \dots + q_{r-1}^2) \times \Big(\sum_{i=-N}^{N-r} x_i\Big)^2. \tag{35}$$

Moreover, if D is the diagonal matrix of size $2N-r+1$ with diagonal entries $0, \dots, 0, q_0/\tilde{q}_0, \dots, q_{r-2}/\tilde{q}_{r-2}$, then we can write $B_N(q)^T = DB_N(\tilde{q})^T$ and hence,

$$\begin{aligned} |(UT_N(p))^T x|^2 &= |T_N(q)^T x|^2 - 2DT_N(q)^T x \cdot B_N(\tilde{q})^T x + |B_N(\tilde{q})^T x|^2 \\ &= |T_N(q)^T x|^2 - |DT_N(q)^T x|^2 + |DT_N(q)^T x - B_N(\tilde{q})^T x|^2 \\ &\ge |T_N(q)^T x|^2 - |DT_N(q)^T x|^2 \\ &\ge \Big(\min_{0\le i\le r-2} 1 - \frac{q_i^2}{\tilde{q}_i^2}\Big)|T_N(q)^T x|^2. \end{aligned} \tag{36}$$

The non-zero constant

$$\min_{0\le i\le r-1} 1 - \frac{q_i^2}{\tilde{q}_i^2} = \min_{0\le i\le r-2} \frac{q_{r-1}^2}{q_{r-1}^2 + (r-1)q_i^2} \tag{37}$$

is independent of N and depends on the polynomials q and p, only.

Regardless of the degree r of $p(z)$ we established the existence of a constant C_4 dependent on $p(z)$ and independent of N such that for any $x \in \mathbb{R}^{2N-r+1}$

$$|(UT_N(p))^T x|^2 \ge C_4 |T_N(q)^T x|^2. \tag{38}$$

After taking the infimum over $x \in \mathbb{R}^{2N-r+1}$ in both sides of the inequality (38) and considering that both matrices $UT_N(p)(UT_N(p))^T$ and $T_N(q)T_N(q)^T$ are symmetric we obtain that

$$\lambda_{min}(UT_N(p)(UT_N(p))^T) \geq \lambda_{min}(T_N(q)T_N(q)^T). \tag{39}$$

Now we combine inequalities (31), (39) and Proposition 1 for $q(z)$, a polynomial whose maximal order of its $\mathbb{T}$-roots is $\alpha - 1$, to obtain (28).

The inequality (27) contains the term $|UT_N(p)\nabla g_0|^2$, that simplifies to

$$\begin{aligned} |UT_N(p)\nabla g_0|^2 &= |U\xi^{2N+1}|^2 = \sum_{i=-N}^{N-r} \left(\xi_i^{2N+1} + \cdots + \xi_{N-r}^{2N+1}\right)^2 \\ &= \sum_{j=1}^{2N-r+1} E\Big[\tau^{N-r-j}\left(\tau^1\Xi + \cdots + \tau^j\Xi\right)\Big| x_{-N}, ..., x_N\Big]^2. \end{aligned} \tag{40}$$

Denote by $s_N(\xi) = \tau^1\Xi + \cdots + \tau^N\Xi$. From L^2-ergodic theorem, [4], for the unitary shift operator τ we have that $\lim_{n\to\infty} \|s_N(\Xi)/N\|_2^2 = 0$. Let $n_\epsilon \geq 1$ be such that $\|s_k(\Xi)/k\|_2^2 < \epsilon$ for any $k \geq N_\epsilon$ and C be such that $\|s_k(\Xi)/k\|_2^2 < C_5$ for any $k \geq 1$. Then

$$\begin{aligned} &\frac{1}{N}\left(\left\|\frac{s_1(\Xi)}{N}\right\|_2^2 + \cdots + \left\|\frac{s_N(\Xi)}{N}\right\|_2^2\right) \\ &= \frac{1}{N}\left(\frac{1}{N^2}\left\|\frac{s_1(\Xi)}{1}\right\|_2^2 + \cdots + \frac{N_\epsilon^2}{N^2}\left\|\frac{s_{N_\epsilon}(\Xi)}{N_\epsilon}\right\|_2^2\right. \\ &\quad \left. + \frac{(N_\epsilon+1)^2}{N^2}\left\|\frac{s_{N_\epsilon+1}(\Xi)}{N}\right\|_2^2 + \cdots + \frac{N^2}{N^2}\left\|\frac{s_N(\Xi)}{N_\epsilon}\right\|_2^2\right) \\ &\leq (\epsilon(N - N_\epsilon) + C_5 N_\epsilon)/N \\ &\leq \epsilon + (C_5 - \epsilon)N_\epsilon/N. \end{aligned} \tag{41}$$

Hence, if $N > C_5 N_\epsilon/\epsilon$ then $\|s_N(\Xi)/N\|_2^2 < 2\epsilon$. Since ϵ was arbitrary we conclude that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N}\left(\left\|\frac{s_1(\Xi)}{N}\right\|_2^2 + \cdots + \left\|\frac{s_N(\Xi)}{N}\right\|_2^2\right) \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{R}^\infty} \left(\frac{\tau^1\Xi + \cdots + \tau^j\Xi}{N}\right)^2 d\mu^\infty = 0. \end{aligned} \tag{42}$$

An implication of (42), the unitary property of the shift operator τ and results on convergence of martingales is

$$\frac{1}{N}\int\limits_{\mathbb{R}^{2N+1}}\left|\frac{UT_N(p)\nabla g_0}{N}\right|^2 d\mu^{2N+1}$$

$$= \lim_{N\to\infty}\frac{1}{N^3}\int\limits_{\mathbb{R}^{2N+1}}\sum_{i=-N}^{N-r}\left(\xi_i^{2N+1}+\cdots+\xi_{N-r}^{2N+1}\right)^2 d\mu^{2N+1}$$

$$= \lim_{N\to\infty}\frac{1}{N^3}\int\limits_{\mathbb{R}^{\infty}}\sum_{i=-N}^{N-r}\left(\tau^i\Xi+\cdots+\tau^{N-r}\Xi\right)^2 d\mu^{\infty}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{2N-r+1}\int\limits_{\mathbb{R}^{\infty}}\left(\frac{\tau^1\Xi+\cdots+\tau^j\Xi}{N}\right)^2 d\mu^{\infty}=0. \tag{43}$$

The conclusion (26) of the corollary follows. □

The systems, (2), in the sequence considered in Varadhan's work, [10], have coefficient matrices $A = T_N(1-z)$ generated by the sequence $p = (1,-1)$ or the polynomial $p(z) = -z+1$. In this case, the maximal order of the $\mathbb{T}$-roots of $p(z)$ is $\alpha = 1$ and the dominating power, N^3, for the L^2-norms of the special solutions, shown by our Corollary 1 and 2, is the power previously established in [10].

Other cases are relevant for results in particle systems. If $p = 1$ and $p(z) = 1$, the systems (4) with coefficient matrices $A = T_N(1)$, would arise from the study of Glauber particle system, a system without interactions. The constant polynomial $p(z) = 1$ has no $\mathbb{T}$-roots, and hence $\alpha = 0$. In this case Corollary 1 establishes the dominating power N for the L^2-norms of the special solutions. If $p = (1,-2,1)$ and $p(z) = 1-2z+z^2 = (z-1)^2$, the systems (4) with coefficient matrices $A = T_N(1-2z+z^2)$, would arise from the study of the continuum solid-on-solid model [8], a particle system with complex interaction. In this case, $p(z)$ has one $\mathbb{T}$-root of order 2, and hence $\alpha = 2$. The dominating powers of Corollary 1 and 2 for the solutions' L^2-norms are N^5 and hence, are too large to establish the scaling limit of the continuum, solid-on-solid model with a non-Gaussian, equilibrium measure, if only the techniques from Varadhan's work are used. This shows the challenges posed by generalizing Varadhan's work to particle systems with complex interactions.

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