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On Characterization of Universal Centers of ODEs with Analytic Coefficients

Alexander Brudnyi

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Abstract. We present a solution of the problem of characterization of the universal centers of a differential equation $v' = \sum_{j=1}^{n} a_j v^{j+1}$ with all a_j real analytic in a neighbourhood of $[a,b] \Subset \mathbb{R}$ in terms of the vanishing of finitely many moments determined by $a_1, \dots, a_n$.

Résumé. On présente la solution du problème de caractériser les centres universels d'une équation différentielle $v' = \sum_{j=1}^{n} a_j v^{j+1}$ dont tous les coefficients sont des fonctions analytiques réelles autour de $[a,b] \Subset \mathbb{R}$ en utilisant les ensembles des zéros d'un nombre fini des moments calculés en partant des fonctions $a_1, \dots, a_n$.

1. Introduction

Let $\tilde{a}_1, \dots, \tilde{a}_n$ be nonconstant real Lipschitz functions on an interval $[a,b] \Subset \mathbb{R}$.

Definition 1.1. *A moment of degree d is an expression of the form*

$$\int_a^b \tilde{a}_1(t)^{d_1} \cdots \tilde{a}_n(t)^{d_n} \cdot \tilde{a}_i'(t)\, dt, \tag{1}$$

where $1 \le i \le n$, all $d_j \in \mathbb{Z}_+$ and $\sum_{j=1}^{n} d_j = d$.

Moments play an important role in the study of the *center problem* for a differential equation

$$\frac{dv}{dt} = \sum_{j=1}^{n} a_j v^{j+1}, \quad \text{where all} \quad a_j \in L^\infty([a,b]), \quad n \in \mathbb{N} \cup \{\infty\}, \tag{2}$$

see [AL], [BlRY], [BFY], [BRY], [BY], [B1]–[B6], [C], [CGM1]–[CGM3], [CL], [FPYZ], [GGL], [MP], [P1], [PRY], [P2], [Z] and references therein, formulated as follows:

Given an equation (2), *determine whether it has a center on* $[a,b]$.

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Recall that equation (2) has a *center* on $[a,b]$ if for all sufficiently small initial values v_0 the corresponding solution satisfies $v(b) = v(a) := v_0$. (Note that the classical Poincaré center-focus problem for polynomial vector fields can be equivalently reformulated as the center problem for equations (2) with all a_j being trigonometric polynomials of a special form.)

The simplest and, in a certain statistical sense, the most frequently occurring centers, the so-called *universal centers*, are defined by the vanishing of all iterated integrals in the coefficients $a_1, a_2, \ldots$. In this case, all possible moments in variables $\tilde{a}_i(t) := \int_a^t a_i(s)\,ds$, $t \in [a,b]$, $i = 1,2,\ldots$, are zeros. Conversely, as follows from [B6, Th. 1.2] if all a_i are real analytic in a neighbourhood of $[a,b]$, then the vanishing of all moments in functions $\tilde{a}_i$ implies that the corresponding equation (2) determines a universal center.

While the structure of universal centers of equation (2) is well understood (see, e.g., Theorem 2.2 below), all other centers of this equation are of an obscure nature requiring further scrutiny.

In this paper we study the following *restricted version of the center problem*:

Given an equation (2)*, determine whether it has a universal center on* $[a,b]$.

We present a solution of this problem for equations (2) with analytic coefficients in terms of the vanishing of finitely many moments in the corresponding functions $\tilde{a}_i$. Moreover, our solution can be realized as a relatively simple algorithm.

For instance, let us describe our approach for an Abel equation

$$\frac{dv}{dt} = a_1 v^2 + a_2 v^3 \tag{3}$$

with a_1, a_2 real analytic in a neighbourhood of $[a,b]$.

To determine whether this equation has a universal center on $[a,b]$ we proceed as follows (see Theoreom 3.3 below):

1. Draw the image $\Gamma_{\tilde{A}}$ of the plane curve
$$\tilde{A} := (\tilde{a}_1, \tilde{a}_2) : [a,b] \to \mathbb{R}^2, \quad \text{where} \quad \tilde{a}_i(t) := \int_a^t a_i(s)ds, \quad i = 1,2.$$
Since the necessary condition for equation (3) to have a center on $[a,b]$ is $\tilde{a}_i(a) = \tilde{a}_i(b)$, $i = 1,2$, we may assume without loss of generality that $\tilde{A}$ is a closed curve.
2. If $\Gamma_{\tilde{A}}$ does not contain subsets homeomorphic to a circle, then the corresponding equation (3) has a universal center on $[a,b]$.
3. For otherwise, measure the diameter $d_{\Gamma_{\tilde{A}}}$ of $\Gamma_{\tilde{A}}$ and compute the maximum of areas of bounded connected components of the complement $\mathbb{R}^2 \setminus \Gamma_{\tilde{A}}$ (denoted by $\mathcal{A}_{\Gamma_{\tilde{A}}}$), and the minimum of sidelengths of maximal squares (with sides parallel to the coordinate axes) inscribed in each of these components (denoted by $r_{\Gamma_{\tilde{A}}}$).

4. Define the number $N_{\Gamma_{\tilde{A}}} \in \mathbb{N}$ by the formula

$$N_{\Gamma_{\tilde{A}}} := \left\lfloor \frac{27\pi}{2} \cdot \frac{\mathcal{A}_{\Gamma_{\tilde{A}}} d_{\Gamma_{\tilde{A}}}}{r^3_{\Gamma_{\tilde{A}}}} \right\rfloor + 1.$$

5. Compute all moments of degree $\leq N_{\Gamma_{\tilde{A}}}$ in the variables $\tilde{a}_1$ and $\tilde{a}_2$.
6. Equation (3) has a universal center on $[a,b]$ if and only if all such moments are equal to zero.

Similarly, there exists an effective algorithm solving the restricted version of the center problem for equations (2) with analytic coefficients $a_1, \dots, a_n$, $3 \leq n < \infty$, in terms of certain geometric characteristics of the image $\Gamma_{\tilde{A}}$ of the curve

$$\text{(4)} \quad \tilde{A} := (\tilde{a}_1, \dots, \tilde{a}_n) : [a,b] \to \mathbb{R}^n, \quad \text{where} \quad \tilde{a}_i(t) := \int_a^t a_i(s)ds, \quad 1 \leq i \leq n,$$

see subsection 4.3 below. Also, we show that this problem can be reduced to the case $n = 2$, see Theorem 4.1 below.

Note that there exist rectifiable curves $\Gamma \Subset \mathbb{R}^n$ satisfying the property that if an equation (2) has a center on $[a,b]$ and $\Gamma_{\tilde{A}} \subset \Gamma$, then this center must be universal. We call such curves Γ *universal*. The basic examples of universal curves in $\mathbb{R}^n$ are *rectangular curves* consisting of finitely many intervals each parallel to one of the coordinate axes, see [B4], or rectifiable triangulable curves in $\mathbb{R}^2$ whose fundamental groups are either trivial or isomorphic to $\mathbb{Z}$, see [B1]. In a forthcoming paper we will establish the following result:

If $\Gamma \Subset \mathbb{R}^2$ is a rectifiable triangulable universal curve and $H : [0,1] \times \Gamma \to \mathbb{R}^2$ is a Lipschitz isotopy analytic with respect to the first coordinate varying in $(0,1)$ and such that $H(0,\cdot) = \mathrm{id}|_\Gamma$, then there exists at most countable subsets $S \subset [0,1]$ so that for each $t \in [0,1] \setminus S$ the curve $H(t,\Gamma) \Subset \mathbb{R}^2$ is universal.

Deforming plane rectangular curves by such isotopies, one can construct sufficiently many smooth and analytic universal curves in $\mathbb{R}^2$.

The proofs of all subsequent results can be found in [B7].

2. Equivalent definitions of universal centers

2.1. Lipschitz triangulable curves We will consider the general equation (2) with $n \in \mathbb{N}$ and all $a_i \in L^\infty([a,b])$. Then the corresponding map $\tilde{A}$ given by (4) is Lipschitz. We will assume that functions a_i are such that

(*) The image $\Gamma_{\tilde{A}}$ of $\tilde{A}$ is a *Lipschitz triangulable curve*.

The latter means that $\Gamma_{\tilde{A}}$ can be presented as the union of finitely many arcs γ_i such that for $i \neq j$ the intersection $\gamma_i \cap \gamma_j$ is either empty or consists of at most one of their endpoints, each γ_i is parameterized by a Lipschitz map $h_i : [0,1] \to \gamma_i \subset \mathbb{R}^n$ such that the inverse map $h_i^{-1} : \gamma_i \to [0,1]$ is locally Lipschitz apart from the endpoints.

The class of Lipschitz triangulable curves includes, in particular, piecewise C^1 curves and images of nonconstant analytic maps $[a,b] \to \mathbb{R}^n$, see, e.g., [BY, Ex. 5.1].

According to the definition, $\Gamma_{\tilde{A}}$ is a finite one-dimensional CW-complex and so it is homotopically equivalent to the wedge sum of circles. In particular, the fundamental group $\pi_1(\Gamma_{\tilde{A}})$ of $\Gamma_{\tilde{A}}$ with base point $0 \in \mathbb{R}^n$ is a finitely generated free group.

2.2. Traversable and unicursal curves An *Eulerian trail* in an undirected connected graph is a path that uses each edge exactly once. If such a path exists, the graph is called *traversable*. If, in addition, it is closed, the graph is called *unicursal*.

It is well known that a connected undirected graph is traversable if and only if at most two of its vertices have odd degree and is unicursal if every its vertex has even degree.

A connected one-dimensional CW-complex X is said to be traversable or unicursal if it is homeomorphic to a traversable or unicursal graph. In this case, an Eulerian trail $\gamma : I \to X$, where I is a closed arc of the unit circle $\mathbb{S}$, is a path that uses each 1-cell of X exactly once. (If X is unicursal, then γ is closed, i.e., $I = \mathbb{S}$). We say that a path $\tilde{\gamma} : [a,b] \to X$ covers γ if there exists a path $\gamma' : [a,b] \to I$ such that $\tilde{\gamma} = \gamma \circ \gamma'$.

The next result is a straightforward corollary of [B6, Th. 1.5].

THEOREM 2.1. *Let $F : [a,b] \to \mathbb{R}^n$ be a nonconstant analytic map defined in a neighbourhood of $[a,b]$. Then the Lipschitz triangulable curve $\Gamma_F := F([a,b])$ is traversable. Moreover, $F : [a,b] \to \Gamma_F$ covers an Eulerian trail in Γ_F.*

In addition to (*) we assume that equation (2) satisfies the following condition:

(**) $\Gamma_{\tilde{A}}$ is traversable and $\tilde{A}$ covers an Eulerian trail in $\Gamma_{\tilde{A}}$.

The class of equations (2) satisfying conditions (*) and (**) contains, in particular, equations with analytic coefficients (but, in fact, it is much wider).

2.3. Equivalent definitions of universal centers In our approach we use the following result.

THEOREM 2.2. *Let $F = (f_1, \dots, f_n) : [a,b] \to \mathbb{R}^n$ be a Lipschitz map such that Γ_F is Lipschitz triangulable and traversable and F covers an Eulerian trail in Γ_F. The following conditions are equivalent:*

(a) *The path $F : [a,b] \to \Gamma_F$ is closed (i.e., $F(a) = F(b)$) and represents the unit element of $\pi_1(\Gamma_F)$ (i.e., it is contractible in Γ_F);*

(b) *F satisfies the composition condition, i.e., $F = \tilde{F} \circ \tilde{f}$ for continuous maps $\tilde{F} : [a, b] \to \Gamma_F$ and $\tilde{f} : [a, b] \to [a, b]$ such that $\tilde{f}(a) = \tilde{f}(b)$;*

(c) *For all possible $i_1, \dots, i_k \in \{1, \dots, n\}$ the iterated integrals*

$$\int \cdots \int_{a \le t_1 \le \cdots \le t_k \le b} f'_{i_k}(t_k) \cdots f'_{i_1}(t_1)\, dt_k \cdots dt_1$$

are equal to zero;

(d) *The path $F : [a, b] \to \Gamma_F$ is closed and represents an element of the commutator subgroup $[\pi_1(\Gamma_F), \pi_1(\Gamma_F)] \subset \pi_1(\Gamma_F)$;*

(e) *For all possible $d_1, \dots, d_n \in \mathbb{Z}_+$ and $i \in \{1, \dots, n\}$ the moments*

$$\int_a^b f_1(t)^{d_1} \cdots f_n(t)^{d_n} f'_i(t)\, dt$$

are equal to zero.

Equivalence of (a) and (c) for Lipschitz maps F with Lipschitz triangulable images Γ_F is established in [B1, Cor. 1.12, Th. 1.14]. In this setting, these conditions are equivalent to a weaker condition, the, so-called, *tree composition condition*, see [BY, Cor. 3.5],

(b′) *There exist a finite tree T and continuous maps $\tilde{F} : T \to \Gamma_F$ and $\tilde{f} : [a, b] \to T$ with $\tilde{f}(a) = \tilde{f}(b)$ such that $F = \tilde{F} \circ \tilde{f}$.*

Thus, (b) implies (a) and (c).

Clearly, (a) implies (d). Equivalence of (d) and (e) for Lipschitz maps with Lipschitz triangulable images is proved in [B2, Cor. 3.11]. Finally, in [B6, Sec. 3.1] the implication (d)$\Rightarrow$(b) is proved for F analytic. The proof in the general case goes along the same lines and is described in [B7, Sec. 9].

Note that if $\pi_1(\Gamma_F)$ is a free group with at least two generators, then one can easily construct a closed Lipschitz path $[a, b] \to \Gamma_F$ which represents an element of $[\pi_1(\Gamma_F), \pi_1(\Gamma_F)]$ but is not contractible. Thus, (d)$\not\Rightarrow$(a) for a generic F.

Theorem 2.2 implies that *equation* (2) *with conditions (*) and (**) has a universal center on $[a, b]$ if the map $\tilde{A}$ satisfies one of the equivalent conditions (a), (b′) or (c).*

3. Universal centers for Abel equations Our first result is valid for a general Abel equation

$$\frac{dv}{dt} = a_1 v^2 + a_2 v^3 \tag{5}$$

with a_1, a_2 satisfying condition (*) only (see [B1] for its proof).

Theorem 3.1. *Suppose $\tilde{A} : [a, b] \to \mathbb{R}^2$ is a closed path satisfying condition (*) and $\Gamma_{\tilde{A}}$ does not contain subsets homeomorphic to a circle. Then the corresponding Abel equation has a universal center on $[a, b]$.*

Next, assume that the fundamental group $\pi_1(\Gamma_{\tilde{A}})$ (of loops in $\Gamma_{\tilde{A}}$ with base point $0 \in \mathbb{R}^2$) is isomorphic, for some $m \in \mathbb{N}$, to the free group with m generators. In what follows, we will also assume that the moments of degree zero in $\tilde{a}_1, \tilde{a}_2$ vanish, or, equivalently, that

$$\tilde{A}(a) = \tilde{A}(b) = 0. \tag{6}$$

(This is the necessary condition for equation (5) to have a center on $[a, b]$.)

Let $\ell_1, \dots, \ell_m : [0, 1] \to \Gamma_{\tilde{A}}$ be simple closed Lipschitz curves in $\mathbb{R}^2$ considered with counterclockwise orientation such that their images $[\ell_1], \dots, [\ell_m]$ in the homology group $H_1(\Gamma_{\tilde{A}})$ are generators of this group. By the Jordan Theorem images of curves ℓ_i in $\mathbb{R}^2$ are boundaries of some simply connected domains $D_i \subset \mathbb{R}^2$, $1 \le i \le m$. Let $\mathcal{A}(D_i)$ stand for the area of D_i.

The following result solves the center problem for a generic equation (5) satisfying conditions (*) and (**) with $\Gamma_{\tilde{A}}$ homotopic to the wedge sum of m circles (see [B7, Th. 4.4] for the proof).

THEOREM 3.2. *Suppose that $\mathcal{A}(D_1), \dots, \mathcal{A}(D_m)$ are linearly independent over the field $\mathbb{Q}$ of rational numbers. Then equation* (5) *corresponding to the closed path $\tilde{A} : [a, b] \to \Gamma_{\tilde{A}}$ satisfying conditions (*), (**) has a center on $[a, b]$ if and only if*

$$\int_a^b \tilde{a}_1(t) \cdot a_2(t)\, dt = 0.$$

Moreover, this center is universal.

Further, let us describe the solution of the restricted version of the center problem for nongeneric equations (5) with $\Gamma_{\tilde{A}}$ as above. To this end, consider the family $\{S_j\}_{1 \le j \le m}$ of bounded connected components of the open set

$$S = \mathbb{R}^2 \setminus \Gamma_{\tilde{A}}.$$

Let r_j be the sidelength of the maximal open square with sides parallel to the coordinate axes contained in S_j. We set

$$r_{\Gamma_{\tilde{A}}} := \min_{1 \le j \le m} \{r_j\}.$$

By $\mathcal{A}_j$ we denote the area of S_j and set

$$\mathcal{A}_{\Gamma_{\tilde{A}}} := \max_{1 \le j \le m} \{\mathcal{A}_j\}.$$

Finally, by $d_{\Gamma_{\tilde{A}}}$ we denote half of the sidelength of the minimal closed square with sides parallel to the coordinate axes containing S.

Let

$$N_{\Gamma_{\tilde{A}}} := \left\lfloor \frac{27\pi}{2} \frac{A_{\Gamma_{\tilde{A}}} d_{\Gamma_{\tilde{A}}}}{r^3_{\Gamma_{\tilde{A}}}} \right\rfloor + 1. \tag{7}$$

One can easily check that $\sqrt{m} \le \frac{d_{\Gamma_{\tilde{A}}}}{r_{\Gamma_{\tilde{A}}}} < N_{\Gamma_{\tilde{A}}}$.

Theorem 3.3. *Equation* (5) *satisfying conditions (*) and (**) has a universal center on* $[a, b]$ *if and only if all moments*

$$\int_a^b \tilde{a}_1(t)^{d_1} \cdot \tilde{a}_2(t)^{d_2} \cdot a_2(t)\, dt \quad \textit{with} \quad \max\{d_1 - 1, d_2\} \le N_{\Gamma_{\tilde{A}}}$$

are equal to zero.

4. Universal centers for general ODEs

4.1. Reduction to Abel equations First, we will show how to reduce the restricted center problem for ODEs

$$\frac{dv}{dt} = \sum_{j=1}^{n} a_j v^{j+1}, \qquad t \in [a, b], \quad 3 \le n < \infty, \tag{8}$$

satisfying conditions (*) and (**) to that for Abel equations.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^n$. For the map $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_n) \to \mathbb{R}^n$ defined by equation (8) (see (4)) and $w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n$, we denote the Lipschitz map that sends $t \in [a, b]$ to a point in $\mathbb{R}^2$ with coordinates $(\langle w_1, \tilde{A}(t) \rangle, \langle w_2, \tilde{A}(t) \rangle)$ by $\tilde{A}_w : [a, b] \to \mathbb{R}^2$.

The necessary condition for (8) to have a universal center on $[a, b]$ is

$$\tilde{A}(a) = \tilde{A}(b) = 0. \tag{9}$$

The following result is a consequence of [B7, Th. 3.1].

Theorem 4.1. *Equation* (8) *satisfying conditions (*), (**) and* (9) *has a universal center on* $[a, b]$ *if and only if for almost all* $w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n$ *(in the sense of Lebesgue measure) Abel equations*

$$\frac{dv}{dt} = \left\langle w_1, \frac{d\tilde{A}}{dt} \right\rangle v^2 + \left\langle w_2, \frac{d\tilde{A}}{dt} \right\rangle v^3 \tag{10}$$

have universal centers on $[a, b]$.

Remark 4.2. (1) Given equation (8) we describe explicitly the set of points $w \in \mathbb{R}^n \times \mathbb{R}^n$ for which Theorem 4.1 is valid.

(2) If equation (8) satisfying conditions (*), (**) and (9) is such that for almost all $w \in \mathbb{R}^n \times \mathbb{R}^n$ the corresponding equations (10) satisfy conditions (*) and (**), then one can apply the algorithm of Theorem 3.3 to one of these equations to decide whether equation (8) has a universal center on $[a, b]$. This method works if, e.g., $\Gamma_{\tilde{A}} \subset \mathbb{R}^n$ is a piecewise C^1 curve and $\tilde{A}$ and $\Gamma_{\tilde{A}}$ satisfy condition (**) (it is a consequence of the Sard Theorem) or $\tilde{A} : [a, b] \to \mathbb{R}^n$ is analytic.

Theorem 4.1 allows to generalize the main results of [CGM3] to the multidimensional case. Specifically, we have

THEOREM 4.3. *If in equation* (8) *the functions* $a_1, \dots, a_n$ *are either real univariate polynomials on* $[a, b]$ *or trigonometric polynomials on* $[0, 2\pi]$ *of degrees* $\leq d$*, then this equation has a universal center on* $[a, b]$ *or on* $[0, 2\pi]$ *if and only if all moments in* $\tilde{a}_1, \dots, \tilde{a}_n$ *of degrees* $2d-3$ *in the first case and of degrees* $4d-3$ *in the second one vanish.*

For $n = 2$ this result is established in [CGM3] by means of the Lüroth Theorem. In general, using Theorem 4.1 one reduces the statement to the case $n = 2$ (because for each $w \in \mathbb{R}^n \times \mathbb{R}^n$ the corresponding functions in equations (10) are either polynomials or trigonometric polynomials of degrees $\leq d$).

4.2. Other results and methods The following result is the extension of Theorem 3.1. It solves the restricted version of the center problem for $\Gamma_{\tilde{A}}$ homotopically trivial (we refer [B1] for the proof).

THEOREM 4.4. *Suppose* $\tilde{A} : [a, b] \to \mathbb{R}^n$ *is a closed path satisfying condition (*) and* $\Gamma_{\tilde{A}}$ *does not contain subsets homeomorphic to a circle. Then the corresponding equation* (8) *has a universal center on* $[a, b]$.

Now we consider the case of equation (8) satisfying condition (*) for which the fundamental group $\pi_1(\Gamma_{\tilde{A}})$ (of loops in $\Gamma_{\tilde{A}}$ with base point $0 \in \mathbb{R}^n$) is isomorphic, for some $m \in \mathbb{N}$, to the free group with m generators and give a geometric solution of the restricted version of the center problem for curves $\Gamma_{\tilde{A}}$ satisfying an additional property without reduction to Abel equations.

Let $C_1, \dots, C_m \subset \Gamma_{\tilde{A}}$ be images of simple closed Lipschitz curves which generate the homology group $H_1(\Gamma_{\tilde{A}})$. Since $\Gamma_{\tilde{A}}$ is Lipschitz triangulable, there exist Lipschitz curves $h_1, \dots, h_k : [0, 1] \to \cup_{1 \leq j \leq m} C_j =: C$ forming a triangulation $\mathcal{T}$ of C. We set $\gamma_i := h_i([0, 1])$, $1 \leq i \leq k$.

Notation. Below all cubes in $\mathbb{R}^n$ are assumed to have sides parallel to the coordinate axes. The radius of such a cube is half of its sidelength.

Assume that $\Gamma_{\tilde{A}}$ satisfies the following additional property:

(***) There exist families of pairwise disjoint open cubes $K_i \subset \mathbb{R}^n$ with centers $c_i \in \gamma_i$ and natural numbers $s_i \in [1, n]$, $1 \leq i \leq k$, such that for each i

- (a) $K_i \cap \gamma_i$ is connected and $K_i \cap \gamma_j = \varnothing$ for all $j \neq i$;
- (b) the restriction to $K_i \cap \gamma_i$ of the projection $\pi_{s_i} : \mathbb{R}^n \to \mathbb{R}^n$, $\pi_{s_i}\big((x_1, \dots, x_n)\big) := (0, \dots, 0, x_{s_i}, 0, \dots, 0)$, is injective.

(This condition is valid, e.g., if $\Gamma_{\tilde{A}}$ is piecewise C^1, or $\tilde{A} : [a, b] \to \mathbb{R}^n$ is a nonconstant analytic map.)

Let r_i be the radius of K_i and l_i be linear measure of the set $\pi_{s_i}\big(\frac{1}{2}K_i \cap \gamma_i\big)$, where $\frac{1}{2}K_i := \frac{1}{2}(K_i - c_i) + c_i$ (the cube $\frac{1}{2}$-homothetic to K_i with respect to c_i). We set

$$r_{\mathcal{T}} := \min_{1 \leq i \leq k}\{r_i\} \quad \text{and} \quad l_{\mathcal{T}} := \min_{1 \leq i \leq k}\{l_{s_i}\}.$$

(According to condition (***), $l_{\mathcal{T}} > 0$.)

Also, we denote

$$L_i := \int_0^1 \|h_i'(t)\|_{\ell_1^n}\, dt, \qquad L_{\mathcal{T}} = \max_{1\le i\le k} L_i;$$

(Here $\|(x_1,\dots,x_n)\|_{\ell_1^n} := \sum_{j=1}^n |x_j|$.)

Finally, by $d_{\Gamma_{\tilde{A}}}$ we denote the radius of the minimal closed cube containing C.

Let

$$\bar{N}_{\mathcal{T}} := \left\lfloor \frac{2\pi n L_{\mathcal{T}} d_{\Gamma_{\tilde{A}}}}{r_{\mathcal{T}} l_{\mathcal{T}}} \right\rfloor + 1. \tag{11}$$

We define

$$\bar{N}_{\Gamma_{\tilde{A}}} := \min_{\mathcal{T}} \bar{N}_{\mathcal{T}}, \tag{12}$$

where the minimum is taken over all triangulations of $\Gamma_{\tilde{A}}$ satisfying condition (***).

Theorem 4.5. *Equation* (8) *satisfying conditions (*), (**), (***) and* (9) *has a universal center on* $[a,b]$ *if and only if all moments*

$$\int_a^b \tilde{a}_1(t)^{d_1}\cdots\tilde{a}_n(t)^{d_n}\cdot a_i(t)\,dt \quad \textit{with} \quad \max_{1\le j\le n}\{d_j\} \le \bar{N}_{\Gamma_{\tilde{A}}}, \quad 1\le i\le n,$$

are equal to zero.

4.3. Example Suppose that equation (8) is such that $\Gamma_{\tilde{A}} \Subset \mathbb{R}^n$ consists of N intervals I_j, each parallel to a coordinate axis, of lengths l_j, $1\le j\le N$. They form a triangulation $\mathcal{T}$ of $\Gamma_{\tilde{A}}$. For this triangulation, taking c_j as the midpoint of I_j and $r_j := \frac{1}{4}l_j$, $1\le j\le N$, one obtains

$$r_{\mathcal{T}} = \min_{1\le j\le N}\left\{\frac{1}{4}l_j\right\},\ l_{\mathcal{T}} = \min_{1\le j\le N}\left\{\frac{1}{4}l_j\right\},\ L_{\mathcal{T}} = \max_{1\le j\le N}\{l_j\} \text{ and } d_{\Gamma_{\tilde{A}}} \le \frac{1}{2}\sum_{j=1}^N l_j.$$

Hence,

$$\begin{aligned} \bar{N}_{\Gamma_{\tilde{A}}} &\le \left\lfloor \frac{16\pi n\cdot\max_{1\le j\le N}\{l_j\}\cdot\sum_{j=1}^N l_j}{\min_{1\le j\le N}\{l_j^2\}} \right\rfloor + 1 \\ &\le \left\lfloor \frac{16\pi n N\cdot\max_{1\le j\le N}\{l_j^2\}}{\min_{1\le j\le N}\{l_j^2\}} \right\rfloor + 1 =: \widetilde{N}. \end{aligned}$$

(If all l_j are equal, then $\widetilde{N} = \lfloor 16\pi nN \rfloor + 1 \sim N$.)

Therefore, since $\Gamma_{\tilde{A}}$ is a universal curve (see the Introduction), equation (8) satisfying condition (**) with such $\Gamma_{\tilde{A}}$ has a center on $[a,b]$ if and only if all moments

$$\int_a^b \tilde{a}_1(t)^{d_1} \cdots \tilde{a}_n(t)^{d_n} \cdot a_i(t)\, dt \quad \text{with} \quad \max_{1\le j\le n}\{d_j\} \le \widetilde{N}, \quad 1 \le i \le n,$$

are equal to zero.

4.4. Algorithm Condition (***) suggests the following algorithm to determine whether equation (8) with all a_i analytic in a neighbourhood of $[a,b]$ has a universal center on $[a,b]$.

1. Draw the image $\Gamma_{\tilde{A}}$ of the analytic map $\tilde{A} \to \mathbb{R}^n$, see (4) (assuming that $\tilde{A}$ satisfies condition (9), i.e., $\tilde{A}$ is a closed path).
2. If $\Gamma_{\tilde{A}}$ does not contain subsets homeomorphic to a circle, then the corresponding equation (8) has a universal center on $[a,b]$.
3. For otherwise, present $\Gamma_{\tilde{A}}$ as union of finitely many arcs γ_i, $1 \le i \le m$, with pairwise disjoint interiors.
4. Choose smooth points $c_i \in \gamma_i$, $1 \le i \le m$.
5. For each i, choose a coordinate axis x_{s_i}, $1 \le s_i \le n$, and a subarc $\tilde{\gamma}_i$ of γ_i with midpoint c_i such that the orthogonal projection π_{s_i} from $\mathbb{R}^n$ onto x_{s_i} is injective when restricted to $\tilde{\gamma}_i$. (Thus, $\tilde{\gamma}_i$ is the graph of a smooth map defined on the interval $\pi_{s_i}(\tilde{\gamma}_i) \subset x_i$.)
6. Choose pairwise disjoint open cubes K_i with centers at c_i of radii r_i not exceeding half-lengths l_i of the intervals $\pi_{s_i}(\tilde{\gamma}_i) \subset x_i$, $1 \le i \le m$.
7. Compute $r := \min_{1\le i\le m} r_i$, $l = \min_{1\le i\le m} l_i$, the radius d of the minimal closed cube containing $\Gamma_{\tilde{A}}$ and maximum L of lengths of arcs γ_i, $1 \le i \le m$.
8. Compute the natural number
$$\bar{N} := \left\lfloor \frac{2\pi n^{3/2} L d}{rl} \right\rfloor + 1.$$
(By our construction, $\bar{N} \ge \bar{N}_{\Gamma_{\tilde{A}}}$, see (12).)
9. Compute all moments
$$\int_a^b \tilde{a}_1(t)^{d_1} \cdots \tilde{a}_n(t)^{d_n} \cdot a_i(t)\, dt \quad \text{with} \quad \max_{1\le j\le n}\{d_j\} \le \bar{N}, \quad 1 \le i \le n.$$
10. Equation (8) has a universal center on $[a,b]$ if and only if all such moments are equal to zero.

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Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, T2N 1N4
e-mail: albru@math.ucalgary.ca