C. R. Math. Rep. Acad. Sci. Canada Vol. **36** (1) 2014, pp. 1–14

Differential Relations on Weak Markov Sets

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Presented by Pierre Millman, FRSC

Abstract. The concept of a weak Markov set takes its origin from Whitney problems for differentiable functions on $\mathbb{R}^n$. These are the only sets for which differential calculus similar to that for open subsets of $\mathbb{R}^n$ can be developed to some extent. This paper surveys some recent results in this direction obtained by the author. In particular, we show that some classical results for smooth functions and differential forms (such as the Poincaré Lemma, the de Rham and Hartogs theorems, the Künneth formulas, etc.) are valid also on certain weak Markov sets and more generally on certain topological spaces with weak Markov structures. The class of such spaces includes C^∞ manifolds with boundaries and some Lipschitz and fractal topological manifolds.

Résumé. Le concept d'un ensemble faible de Markov provient des problèmes de Whitney pour des fonctions différentiables sur $\mathbb{R}^n$. Ce sont les seuls ensembles pour lesquels le calcul différentiel semblable à celui pour les sous-ensembles ouverts de $\mathbb{R}^n$ peut être développé dans une certaine mesure. Cet article examine quelques résultats récents dans cette direction obtenus par l'auteur. En particulier, on prouve que quelques résultats classiques pour les fonctions lisses et les formes différentielles (comme le lemme de Poincaré, les théorèmes de de Rham et de Hartogs, les formules de Künneth, etc.) sont également valides sur quelques ensembles faibles de Markov et plus généralement sur quelques espaces topologiques munis de structures faibles de Markov. La classe de tels espaces inclut les variétés lisses à bord et quelques variétés topologiques lipschitziennes et fractales.

1. **Introduction** Motivated by boundary value problems for partial differential equations, classical trace and extension theorems characterize traces of spaces of generalized smoothness (e.g., Sobolev, Besov, etc.) to smooth submanifolds of a Euclidean space. But in many cases one needs similar results for subsets of a more complicated geometric structure (for instance, after the change of variables initial data may be situated on a Lipschitz surface). The subject is originated from the seminal 1934 Hassler Whitney papers [W1], [W2] the first of which deals with the following problem: *characterize collections of m-jets on a closed subset of $\mathbb{R}^n$ that are traces of C^m jets, i.e., sets of Taylor polynomials of degree m on $\mathbb{R}^n$ generated by C^m functions.*

Received by the editors on March 26, 2013; revised May 18, 2013.
Research is supported in part by NSERC.
AMS Subject Classification: Primary: 26B05; secondaries: 28A80, 18F15, 14F40.
Keywords: Weak Markov set, Whitney problems, trace, extension, C^k function, de Rham cohomology.

Whitney developed important analytic and geometric techniques which allowed him to solve this problem; moreover, he constructed a linear bounded extension operator acting from the trace space to $C^m(\mathbb{R}^n)$. In the second paper, marked by number I, he used his extension method to solve for univariate functions an essentially more difficult (known as the first Whitney) problem of *characterizing continuous functions on an arbitrary closed subset of* $\mathbb{R}^n$ *that are traces of* $C^m(\mathbb{R}^n)$ *functions.*

In accordance with the one-dimensional case it was natural to formulate the second (Whitney) problem: *Does there exists a linear bounded extension operator from the trace of* $C^m(\mathbb{R}^n)$ *to a closed subset of* $\mathbb{R}^n$ *to* $C^m(\mathbb{R}^n)$*?*

Although in the second half of the 20th century some important partial results were obtained, the problems remained unsolved. In the last decade a significant breakthrough in the area has been made by Charles Fefferman (see, in particular, his survey [F2] and references therein) who solved the second Whitney problem and made an important contribution towards the solution of the first Whitney problem justifying the, so-called, Yu.Brudnyi-Shvartsman finiteness principle for traces of C^m functions to closed subsets of $\mathbb{R}^n$. Fefferman's results are based on multileveled, voluminous constructions that can be turned into algorithms for finite sets.

Inspired by these results, the author and Yuri Brudnyi introduced in [BB2] the class of weak Markov sets for which solutions of Whitney problems are much simpler and invoke the basic ideas from the original Whitney papers. By $\mathcal{P}_{k,n}$ we denote the space of real polynomials on $\mathbb{R}^n$ of degree k and by $Q_r(x) \subset \mathbb{R}^n$ the closed cube centered at x of sidelength $2r$.

DEFINITION 1.1. *A point x of a subset $S \subset \mathbb{R}^n$ is said to be weak k-Markov if*

$$\varliminf_{r\to 0}\left\{\sup_{p\in\mathcal{P}_{k,n}\setminus\{0\}}\left(\frac{\sup_{Q_r(x)}|p|}{\sup_{S\cap Q_r(x)}|p|}\right)\right\}<\infty. \tag{1}$$

A closed set $S \subset \mathbb{R}^n$ is said to be weak k-Markov if it contains a dense subset of weak k-Markov points.

The term *weak Markov set* is derived from that of *Markov set* introduced by Alf Jonsson and Hans Wallin in [JW] by means of local Markov polynomial inequalities in connection with extension and trace problems for functions in Besov spaces, see Section 2 below for an equivalent definition. The class of weak k-Markov sets denoted by $\mathrm{Mar}_k^*(\mathbb{R}^n)$, contains, in particular, the closure of any open set, the Ahlfors p-regular compact subsets of $\mathbb{R}^n$ with $p > n-1$, a wide class of fractals of arbitrary positive Hausdorff measure and closures of unions of any combination of such sets. The class $\mathrm{Mar}_0^*(\mathbb{R}^n)$ (consisting of all nonempty closed subsets of $\mathbb{R}^n$) admits a filtration

$$\mathrm{Mar}_0^*(\mathbb{R}^n)\supset\mathrm{Mar}_1^*(\mathbb{R}^n)\supset\mathrm{Mar}_2^*(\mathbb{R}^n)\supset\cdots\supset\mathrm{Mar}_k^*(\mathbb{R}^n)\supset\cdots.$$

We set

$$\mathrm{Mar}_\infty^*(\mathbb{R}^n):=\bigcap_{k\geq 0}\mathrm{Mar}_k^*(\mathbb{R}^n)$$

and refer to elements of this class as weak ∞-Markov sets.

In the present paper we survey some results obtained by the author in [B] related to a version of the first Whitney problem of *characterizing families of continuous functions satisfying certain differential relations* on weak Markov sets. To this end we develop differential calculus on weak Markov sets first introducing (intrinsically) the notion of a derivative of a function on such a set and then establishing its basic properties similar to those for derivatives of functions in $\mathbb{R}^n$. Next, we consider examples of problems for functions satisfying some differential relations and show, in particular, that many classical results for smooth functions and differential forms (such as the Poincaré Lemma, the de Rham and Hartogs theorems, the Künneth formulas, etc.) are valid also on certain weak Markov sets and more generally on certain topological spaces with weak Markov structures. The class of such spaces include, in particular, C^∞ manifolds with boundaries and some Lipschitz and fractal topological manifolds. Thus the paper offers yet another approach to analysis on fractals, a developing area of modern mathematics that focuses on geometric and dynamical aspects of fractals. (For known approaches and major developments in the area, see, e.g., survey [S] and references therein.)

In a forthcoming paper following ideas of H. Whitney [W4] we introduce integration of differential forms over special "fractal" chains in Mar^*_∞ spaces and establish an analog of the Stokes formula for them.

2. Properties of weak Markov sets All properties except for (6), (10) and (11) proved in [B, Sect. 6] either follow directly from Definition 1.1 or have been established in [BB2] (see also [BB1, Vol. II, Ch. 10]).

1. $\mathrm{Mar}^*_0(\mathbb{R}^n)$ *consists of all nonempty closed sets in* $\mathbb{R}^n$.
2. $\mathrm{Mar}^*_1(\mathbb{R})$ *consists of all infinite closed subsets of* $\mathbb{R}$ *without isolated points.*
3. *If* $S \in \mathrm{Mar}^*_k(\mathbb{R}^n)$ *and* $U \subset S$ *is relatively open, then the closure* $\bar{U} \in \mathrm{Mar}^*_k(\mathbb{R}^n)$.
4. *If* $\{S_i\}_{i\in I} \subset \mathrm{Mar}^*_k(\mathbb{R}^n)$, *then* $\overline{\cup_{i\in I} S_i} \in \mathrm{Mar}^*_k(\mathbb{R}^n)$.
5. *For closed sets* $S_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$, *their product* $S_1 \times S_2 \in \mathrm{Mar}^*_k(\mathbb{R}^{n_1+n_2})$ *if and only if* $S_i \in \mathrm{Mar}^*_k(\mathbb{R}^{n_i})$, $i = 1, 2$.
6. *Let* x *be a weak* k*-Markov point of* $S \subset \mathbb{R}^n$. *Suppose a map* $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, $m \le n$, *is such that the derivative* $D\varphi(x)$ *of* φ *at* x *exists and has rank* m. *Then* $\varphi(x)$ *is a weak* k*-Markov point of* $\varphi(S) \subset \mathbb{R}^m$. *In particular, if the set of such points* x *is dense in* S, *then* $\overline{\varphi(S)} \in \mathrm{Mar}^*_k(\mathbb{R}^m)$.
7. *A closed set* $S \subset \mathbb{R}^n$ *belongs to the class of Markov sets, denoted by* $\mathrm{Mar}(\mathbb{R}^n)$, *if for some constant* $c > 0$ *and every* $x \in S$, $0 < r \le \operatorname{diam} S$ *and* $p \in \mathcal{P}_{1,n}$,

$$\sup_{Q_r(x)} |p| \le c \cdot \sup_{S\cap Q_r(x)} |p|.$$

If $S \in \mathrm{Mar}(\mathbb{R}^n)$, *then the above condition holds for polynomials of every degree, see [JW, Ch.2]. Hence,* $\mathrm{Mar}(\mathbb{R}^n) \subset \mathrm{Mar}^*_\infty(\mathbb{R}^n)$.

A closed set $S \subset \mathbb{R}^n$ is called (Ahlfors) *d-regular* ($0 \le d \le n$) if for every cube $Q_r(x)$ with $x \in S$ and $0 < r < \operatorname{diam} S$

$$c_0 r^d \le \mathcal{H}_d(S \cap Q_r(x)) \le c_1 r^d,$$

where $c_0, c_1 > 0$ are constants independent of x and r and $\mathcal{H}_d$ stands for the Hausdorff d-measure on $\mathbb{R}^n$.

For instance, the classical von Koch snowflake curve is d-regular with $d := \ln 4/\ln 3$, see, e.g., [Fa] for its construction.

Every d-regular compact subset of $\mathbb{R}^n$ with $d > n-1$ is Markov, see [JW, Ch.2]. So *the closure of unions of d_i-regular compact sets $S_i \subset \mathbb{R}^n$ with $d_i > n-1$, $i \in I$, is weak ∞-Markov.*

8. *Fix $k \in \mathbb{N}_+ \cup \{\infty\}$. Every closed subset $S \subset \mathbb{R}^n$ is disjoint union of a (possibly empty) weak k-Markov subset and a set of Hausdorff dimension at most $n-1$.*

In particular, let a closed set $S \subset \mathbb{R}^n$ have a base (in the relative topology) consisting of sets of Hausdorff dimension greater than $n-1$. Then S is weak ∞-Markov. (For otherwise, the nonempty open set $S \setminus S_{\max}$, where $S_{\max}$ is the maximal weak ∞-Markov subset of S, contains an element of the base with Hausdorff dimension greater than $n-1$.)

Using this fact one can prove, e.g., that the graph of the *Weierstrass nowhere differentiable function*

$$\sum_{n=0}^{\infty} a^n \cos(b^n x), \qquad \text{where} \quad 0 < a < 1,\ b > 1,\ ab > 1,$$

is weak ∞-Markov. (It follows from [PU, Th. 4] and estimates in [Ha].)

More generally, it was proved in [PU, Th. 5] that for a fixed $0 < \alpha < 1$ the graph of $\varphi(x) = \sum_{n=0}^{\infty} \beta^{-\alpha n} q(\beta^n x + \theta_n)$ with sufficiently large $\beta > 1$ and arbitrary $\theta_0, \theta_1, \dots$, where q is a Lipschitz function on $\mathbb{R}$ of period one, monotone and nonconstant on a compact interval, satisfies the above base topology assumption (for $n = 2$). Hence, this graph belongs to $\mathrm{Mar}^*_\infty(\mathbb{R}^2)$.

9. *A point x of a subset $S \subset \mathbb{R}^n$ is weak k-Markov if and only if there exist a convergent to 0 sequence of positive numbers $\{r_i\}_{i\in\mathbb{N}}$ and a sequence of finite subsets $F_i \subset S \cap Q_{r_i}(x)$ with* $\operatorname{card} F_i = \dim \mathcal{P}_{k,n}$, $i \in \mathbb{N}$, *such that*

$$\sup_{i\in\mathbb{N}} \left\{ \sup_{p\in\mathcal{P}_{k,n}\setminus\{0\}} \left(\frac{\sup_{Q_{r_i}(x)} |p|}{\sup_{F_i} |p|} \right) \right\} < \infty. \tag{2}$$

The proof is a consequence of the following fact (cf. [BB2, Prop. 3.5]):

Let V be an n-dimensional vector space of bounded real functions on a set X (in particular, $\operatorname{card} X \ge n$*). There exist a finite subset $F \subset X$ of* $\operatorname{card} F = n$ *and a constant $c = c(n) > 0$ such that for every $v \in V$,*

$$\sup_X |v| \le c \cdot \max_F |v|.$$

10. *Unlike Markov sets,* $\mathrm{Mar}^*_{k+1}(\mathbb{R}^n)\setminus \mathrm{Mar}^*_k(\mathbb{R}^n)\neq\varnothing$ *for all* $k\in\mathbb{Z}_+$.

In view of (5) it suffices to establish this property for $n=1$ only. It is done in [B, sect. 6.2] by constructing, for each $k\in\mathbb{Z}_+$, a Cantor-type subset of $\mathbb{R}$ of Hausdorff dimension zero belonging to $\mathrm{Mar}^*_k(\mathbb{R})\setminus\mathrm{Mar}^*_{k+1}(\mathbb{R})$.

11. *Classes* $\mathrm{Mar}^*_k(\mathbb{R})$, $k\in\mathbb{Z}_+\cup\{\infty\}$, *are invariant under bi-Lipschitz maps (i.e., if* $S\in\mathrm{Mar}^*_k(\mathbb{R})$ *and* $\varphi:S\to S'\subset\mathbb{R}$ *is a bi-Lipschitz map, then* $S'\in\mathrm{Mar}^*_k(\mathbb{R})$*), while* $\mathrm{Mar}^*_k(\mathbb{R}^n)$, $n\geq 2$, $k\in\mathbb{N}\cup\{\infty\}$, *are not.*

To prove the second part of the statement for $k\in\mathbb{N}$ we construct a Lipschitz function $\varphi:\mathbb{R}\to\mathbb{R}$ whose graph Γ_φ belongs to $\mathrm{Mar}^*_k(\mathbb{R}^2)\setminus\mathrm{Mar}^*_{k+1}(\mathbb{R}^2)$ (see [B, sect. 6.3]).

3. Traces of differentiable functions to weak Markov sets Results of this section are proved in [B, Sect. 7].

Definition 3.1. *A function* $f:S\to\mathbb{R}$ *is said to have derivatives of order* $\leq m\,(\leq k)$ *at a weak* k*-Markov point* $x\in S\subset\mathbb{R}^n$ *if there exists a polynomial* $T^m_x(f)\in\mathcal{P}_{m,n}$ *such that*

$$\lim_{y\to x}\frac{|f(y)-T^m_x(f)(y)|}{\|y-x\|^m_\infty}=0.$$

If $T^m_x(f)(z):=\sum_{|\alpha|\leq m}\frac{c_\alpha}{\alpha!}(z-x)^\alpha$, $\alpha\in\mathbb{Z}^n_+$, $|\alpha|:=\|\alpha\|_1$, *then* c_α *is called the partial derivative of order* α *at* x *and is denoted as* $D^\alpha_S f(x)$.

(Here $\|\cdot\|_\infty$ *and* $\|\cdot\|_1$ *stand for* ℓ^∞*-norm on* $\mathbb{R}^n$ *and* ℓ^1*-norm on* $\mathbb{Z}^n$*, respectively.)*

Proposition 3.2. *A function* $f:S\to\mathbb{R}$ *has derivatives of order* $\leq m\,(\leq k)$ *at a weak* k*-Markov point* $x\in S\subset\mathbb{R}^n$ *if and only if there exists a function* $\tilde f:\mathbb{R}^n\to\mathbb{R}$ *having derivatives of order* $\leq m$ *at* x *such that* $\tilde f|_S=f$. *Moreover, the Taylor polynomial* $T^m_x(\tilde f)$ *of order* m *at* x *of any such extension* $\tilde f$ *coincides with* $T^m_x(f)$.

Next, we express derivatives of $f:S\to\mathbb{R}$ in intrinsic terms using some analogs of divided differences.

Proposition 3.3. *For a weak* k*-Markov point* $x\in S\subset\mathbb{R}^n$*, and a nonnegative integer number* $m\leq k$ *there exist a convergent to* 0 *sequence of positive numbers* $\{r_i\}_{i\in\mathbb{N}}$*, a sequence of finite subsets* $F^m_i\subset S\cap Q_{r_i}(x)\setminus\{x\}$ *with* $\mathrm{card}\,F_i=\dim\mathcal{P}_{m,n}$*, a sequence of signed measures* μ^α_i, $\alpha\in\mathbb{Z}^n_+$, $|\alpha|\leq m$, *on* F^m_i, $i\in\mathbb{N}$, *such that if a function* $f:S\to\mathbb{R}$ *has derivatives of order* m *at* x*, then*

$$D^\alpha_S f(x)=\lim_{i\to\infty}\int_{F^m_i} f\,d\mu^\alpha_i. \tag{3}$$

Recall that the space $C^{m,\omega}(\mathbb{R}^n)$ is defined by the norm

$$\|f\|_{C^{m,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n}|f| + \max_{|\alpha|=m}\sup_{x\neq y}\frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|_\infty)};$$

here $\omega : (0,\infty) \to (0,\infty)$ is a monotone nondecreasing concave function with $\omega(0+) = 0$. For a subset $S \subset \mathbb{R}^n$, by $C^{m,\omega}(S)$ we denote the trace space of continuous functions on S equipped with the trace norm

$$\|f\|_{C^{m,\omega}(S)} := \inf\{\|g\|_{C^{m,\omega}(\mathbb{R}^n)}\,;\, g|_S = f\}.$$

The following result characterizing functions in $C^{m,\omega}(S)$ is an analog of the classical Whitney-Glaeser theorem (see, e.g., [BB1, Vol. I, Sec. 2.2]).

THEOREM 3.4. *Let $S \subset \mathbb{R}^n$ be weak k-Markov, $k \in \mathbb{N}$. A function $f \in C(S)$ belongs to $C^{m,\omega}(S)$, $m \le k$, if and only if it has derivatives of order $\le m\,(\le k)$ at each weak k-Markov point $x \in S$ and there exists a constant $\lambda > 0$ such that for all weak k-Markov points $x, y \in S$, $z \in \{x, y\}$*

$$\max_{|\alpha|\le m}|D_S^\alpha f(x)| \le \lambda \quad \textit{and}$$

(4)

$$\max_{|\alpha|\le m}\frac{|D^\alpha\big(T_x^m(f) - T_y^m(f)\big)(z)|}{\|x-y\|_\infty^{m-|\alpha|}} \le \lambda\cdot\omega(\|x-y\|_\infty).$$

Moreover,

$$\|f\|_{C^{m,\omega}(S)} \approx \inf\lambda$$

with constants of equivalence depending only on m and n.

Finally, we formulate a result proved in [BB2] related to the Whitney problems for the trace of space $C^{m,\omega}(\mathbb{R}^n)$ to weak k-Markov sets ($k \ge m$).

THEOREM 3.5. *Let $S \subset \mathbb{R}^n$ be weak k-Markov, $k \in \mathbb{N}$, and $m \le k$.*

(a) **Finiteness Principle.** *Suppose that for $f \in C(S)$ its restriction to each subset S' of S of cardinality at most $2\binom{n+m}{n}$ satisfies $\|f|_{S'}\|_{C^{m,\omega}(S')} \le 1$. Then $f \in C^{m,\omega}(S)$ and there exists $c = c(m,n) > 0$ such that $\|f\|_{C^{m,\omega}(S)} \le c$.*

(b) **Finite Depth Simultaneous Extension.** *There exists a linear bounded extension operator $E : C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$,*

$$(Ef)(x) := \begin{cases} \displaystyle\sum_{i=1}^\infty \lambda_i(x) f(x_i) & \text{if} \quad x \in \mathbb{R}^n \setminus S \\ f(x) & \text{if} \quad x \in S; \end{cases}$$

here all λ_i are in $C^\infty(\mathbb{R}^n)$ and have compact supports in $\mathbb{R}^n \setminus S$, all $x_i \in S$ and for each $x \in \mathbb{R}^n \setminus S$ the number of nonzero terms in the above sum is at most $\binom{n+m}{n}\cdot w$, where w is the order of the Whitney cover of $\mathbb{R}^n \setminus S$.

REMARK 3.6. If $S \subset \mathbb{R}^n$ is an arbitrary closed set, then, as it was discovered by Ch. Fefferman [F1], a statement similar to (a) is valid with the finiteness constant bounded by $(1+\dim \mathcal{P}_{m,n})^{3\cdot\dim \mathcal{P}_{m,n}}$. Later this bound was replaced by $2^{\dim \mathcal{P}_{m,n}}$ due to Bierstone-Milman [BM] and, independently and by a different method, Shvartsman [Shv]. A linear bounded extension operator $C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$ in this case was constructed by Ch. Fefferman [F2]. Using a modification of his method Luli [Lu] constructed an extension operator $C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$ of finite depth, cf. (b).

4. Examples of extension problems

DEFINITION 4.1. *A function f on an open subset U of a weak k-Markov set $S \subset \mathbb{R}^n$ is said to belong to the space $C^m(U)$, $m \le k$, if each $x \in U$ has an open neighbourhood $O_x \subset U$ such that $f|_{O_x}$ satisfies the conditions of Theorem 3.4 with a modulus of continuity ω_x. Here $C^\infty(U) := \cap_{\ell=1}^\infty C^\ell(U)$.*

We define $C^m(U, \mathbb{R}^p)$ to be the space of maps $U \to \mathbb{R}^p$ with coordinates in $C^m(U)$.

Using Theorem 3.4 and a smooth partition of unity subordinate to a suitable cover of U by open subsets of $\mathbb{R}^n$, one easily shows that $f \in C^m(U)$ if and only if it is restriction to U of a C^m function defined in an open neighbourhood of U in $\mathbb{R}^n$. In the case $m = \infty$ this follows from a result of Hestens [Hes].

Let $S \in \mathrm{Mar}_k^*(\mathbb{R}^n)$ and $S_0 \subset S$ be a dense subset of weak k-Markov points of S. For each $f \in C^m(S)$, $m \le k$, the function $D_S^\alpha f \in C(S)$, $|\alpha| \le m$, is defined as the extension by continuity to S of the function $D_S^\alpha(f|_{S_0})$ (which is locally uniformly continuous on S_0 due to Theorem 3.4). Next, for a differential operator $P_D := \sum_{|\alpha|\le m} a_\alpha D^\alpha$, $a_\alpha \in C(W)$, where $W \subset \mathbb{R}^n$ is an open neighbourhood of S, we define

$$P_{D_S} f := \sum_{|\alpha|\le m} (a_\alpha|_S) D_S^\alpha f. \tag{5}$$

If f is the restriction to S of a C^m function $\hat{f}$ defined in an open neighbourhood of S in W, then (see Proposition 3.2)

$$P_{D_S} f = (P_D \hat{f})|_S. \tag{6}$$

For a subset O of a Hausdorff topological space X, $\partial O := \bar{O} \setminus O^o$ and $O^c := X \setminus O$ stand for its boundary and complement in X.

In the formulated examples we deal with Cantor-type sets of the form

$$S := \bar{\Omega} \setminus \big(\cup_{j\ge 1} D_j\big), \tag{7}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ and $\{D_j\}_{j\ge1}$ is a family of mutually disjoint domains in Ω such that $\partial D_j = \partial(\bar{D}_j)^c$ for all j.

For instance, the Sierpiński gasket is a d-regular set, $d := \ln 3/\ln 2$, of the form (7) obtained by repeatedly removing open equilateral triangles from an initial equilateral triangle, see, e.g., [Fa].

Our first example describes traces of polynomials to a class of Cantor-type weak Markov sets.

THEOREM 4.2. *Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a weak k-Markov set, $k \geq 2$, of the form (7) such that $\Omega \Subset \mathbb{R}^n$ and all $(\bar{D}_j)^c$ are domains. Suppose $f \in C^{m+1}(S)$, $m \leq k-1$, satisfies $D_S^\alpha f = 0$ for all $|\alpha| = m$. Then there exists a unique polynomial $p_f \in \mathcal{P}_{m-1,n}$ such that $p_f|_S = f$. In particular, S is connected.*

Next we present a Hartogs-type theorem describing traces of holomorphic functions to a class of Cantor-type weak Markov sets.

THEOREM 4.3. *Let $S \subset \mathbb{C}^n$, $n \geq 2$, be a weak k-Markov set, $k \geq 2$, of the form (7) such that $\Omega \Subset \mathbb{C}^n$ and all $(\bar{D}_j)^c$ are domains. Let $f \in C^2_{\mathbb{C}}(S) := C^2(S) \otimes \mathbb{C}$ satisfy $\overline{\partial}_S f = 0$. Then there exists a unique holomorphic function $F \in \mathcal{O}(\Omega) \cap C^1_{\mathbb{C}}(\bar{\Omega})$ such that $F|_S = f$ and $\sup_{\bar{\Omega}} |F| = \sup_S |f|$.*

Our third example describes solutions of Laplace-type equations on a class of Cantor-type weak Markov sets. In its formulation $C^{2,\alpha}(\mathbb{R}^n)$ stands for the space of bounded C^2 functions on $\mathbb{R}^n$ whose second-order derivatives satisfy the Hölder condition with exponent α.

THEOREM 4.4. *Let $S \subset \mathbb{R}^n$, $n \geq 3$, be a weak k-Markov set, $k \geq 2$, of the form (7). Let $f \in C^{2,\alpha}(S)$, $0 < \alpha < 1$, satisfy $\Delta_S f = 0$. Then there exist functions $F_1, F_2 \in C^{2,\alpha}(\mathbb{R}^n)$ harmonic in $\cup_{j\geq 1} D_j$ and $\left(\overline{\cup_{j\geq 1} D_j}\right)^c$, respectively, such that $\lim_{x\to\infty} F_i(x) = 0$, $i = 1, 2$, and $(F_1+F_2)|_S = f$. Moreover, if $F_1', F_2' \in C^{2,\alpha}(\mathbb{R}^n)$ is any other pair of such functions, then $F_1' = F_1$ on $\cup_{j\geq 1} D_j$ and $F_2' = F_2$ on $\left(\overline{\cup_{j\geq 1} D_j}\right)^c$.*

5. Spaces with weak Markov structure

5.1. Basic definitions and properties Most definitions of this subsection are similar to those of the theory of standard C^k manifolds, see, e.g., [Mu].

DEFINITION 5.1. *A weak k-Markov structure on a second countable Hausdorff space S is a collection of pairs $\mathcal{D} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, called charts, such that*

1. *$(U_\alpha)_{\alpha\in\Lambda}$ is an open cover of S;*
2. *Each φ_α is a homeomorphism of U_α onto a relatively open subset of a set in $\mathrm{Mar}_k^*(\mathbb{R}^n)$ such that for all $\alpha, \beta \in \Lambda$ the maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)\, (\subset \mathbb{R}^n)$ are of class C^k;*
3. *$\mathcal{D}$ is maximal with respect to property (2).*

A function f on an open subset $U \subset S$ is of class C^k if $f\circ\varphi_\alpha \in C^k(\varphi_\alpha(U\cap U_\alpha))$ for all $(U_\alpha, \varphi_\alpha) \in \mathcal{D}$.

The definition implies that S is paracompact and of covering dimension at most n. Moreover, it has a base of topology consisting of relatively compact open subsets.

The class of spaces with weak k-Markov structure will be denoted Mar_k^*. If M is a C^k manifold, then by $\mathrm{Mar}_k^*(M) \subset \mathrm{Mar}_k^*$ we denote the family of all weak k-Markov closed subsets $S \subset M$, i.e., such that in Definition 5.1 each $(U_\alpha, \varphi_\alpha)$ has the form $U_\alpha := V_\alpha \cap S$ and $\varphi_\alpha := \psi_\alpha|_{U_\alpha}$, where (V_α, ψ_α) is a C^k chart on M. In particular, every C^k manifold and weak k-Markov subset of $\mathbb{R}^n$ belong to Mar_k^*.

Proposition 5.2. *Let $S \in \mathrm{Mar}_k^*$. For any open cover of S there exists a C^k partition of unity subordinate to it.*

For $S_1, S_2 \in \mathrm{Mar}_k^*$ a continuous map $f : S_1 \to S_2$ is said to be of class C^k if it is of class C^k in all suitable charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\alpha 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ on S_1 and S_2, respectively (i.e., $\varphi_{\alpha 2} \circ f \circ \varphi_{\beta 1}^{-1} \in C^k\big((\varphi_{\beta 1} \circ f^{-1})(U_{\alpha 2}), \varphi_{\alpha 2}(U_{\alpha 2})\big)\big)$.

Every C^k map $f : S_1 \to S_2$ determines an injective morphism of algebras $f^* : C^k(U) \to C^k(f^{-1}(U))$, $U \subset S_2$ is open (here f^* is the pullback by f).

For $S \subset \mathrm{Mar}_k^*(\mathbb{R}^n)$, $k \geq 1$, the *tangent bundle* $TS \to S$ on S is defined as the restriction to S of the tangent bundle $T\mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$. Clearly, $TS \in \mathrm{Mar}_k(T\mathbb{R}^n)$. Now, for a C^k map $f : S \to \mathbb{R}^p$ the *differential* $Df : TS \to T\mathbb{R}^p$ is defined as the restriction of the differential $D\hat{f} : TU \to T\mathbb{R}^p$, where $\hat{f}$ is a C^k extension of f to an open neighbourhood $U \subset \mathbb{R}^n$ of S. According to Propositions 3.2 and 3.3, the definition of Df does not depend on the choice of the C^k extension $\hat{f}$ and, in intrinsic terms, $Df := \big(f, (D_S^j f_i)_{1 \leq i \leq n,\, 1 \leq j \leq p}\big)$, where $f := (f_1, \dots, f_p)$.

Similarly, if $S \in \mathrm{Mar}_k^*$, then the *tangent bundle* $TS \to S$ is defined as the quotient space of the disjoint union $\bigsqcup_{\alpha \in \Lambda} TU_\alpha$, where $TU_\alpha \to U_\alpha$ is the pullback by φ_α of the bundle $T\overline{\varphi_\alpha(U_\alpha)}$, by means of the following equivalence relation

$$TU_\alpha \ni v \sim w \in TU_\beta \Longleftrightarrow (\varphi_\beta)_*(w) = D(\varphi_\beta \circ \varphi_\alpha^{-1})\big((\varphi_\alpha)_*(v)\big);$$

here $(U_\alpha, \varphi_\alpha) \in \mathcal{D}$ and $(\varphi_\alpha)_* : TU_\alpha \to T\overline{\varphi_\alpha(U_\alpha)}|_{\varphi_\alpha(U_\alpha)}$ is the isomorphism of bundles generated by φ_α. The weak k-Markov structure on TS is given by the collection of charts $\{(TU_\alpha, (\varphi_\alpha)_*)\}_{\alpha \in \Lambda}$.

Next, if $f : S_1 \to S_2$ is a C^k map between $S_1, S_2 \in \mathrm{Mar}_k^*$, then the *differential* is a morphism of bundles $Df : TS_1 \to TS_2$ defined in charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\beta 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ (such that $f(U_{\alpha 1}) \subset U_{\beta 2}$) by the formula

$$Df(v) := (\varphi_{\beta 2})_*^{-1} \left(D(\varphi_{\beta 2} \circ f \circ \varphi_{\alpha 1}^{-1})\big((\varphi_{\alpha 1})_*(v)\big)\right).$$

We say that $f : S_1 \to S_2$ is a C^k *embedding* if f is an injection and Df is injective on each fibre of TS_1. If, in addition, f is a homeomorphism, then it is called a C^k *diffeomorphism*.

Proposition 5.3. *Suppose $S_1, S_2 \in \mathrm{Mar}_k^*$, $k \geq 1$, and $f : S_1 \to S_2$, is a C^k diffeomorphism. Then $f^{-1} : S_2 \to S_1$ is a C^k diffeomorphism as well.*

Let $S \subset \mathbb{R}^p$ be a closed subset and $S \in \mathrm{Mar}_k^*$. We say that the weak k-Markov structure $\mathcal{D}$ on S is induced from $\mathbb{R}^p$ if each φ_α in Definition 5.1 for S is the restriction of a C^k map into $\mathbb{R}^n$ defined in an open neighbourhood of U_α in $\mathbb{R}^p$.

Our next result is an analog of the Whitney embedding theorem [W3].

PROPOSITION 5.4. *Suppose $S \in \mathrm{Mar}_k^*$, $k \geq 1$, and the rank of TS is n. There exists a proper C^k embedding $f : S \to \mathbb{R}^N$, where $N := 2n+1$ if $k \geq 2$, such that $f(S) \in \mathrm{Mar}_k^*$ and has the weak k-Markov structure induced from $\mathbb{R}^N$. Moreover, $f : S \to f(S)$ is a C^k diffeomorphism.*

(In case $S \in \mathrm{Mar}_k^*(M)$, $k \geq 2$, where $\dim M = n$, one can replace $2n+1$ by $2n$ by applying the strong Whitney embedding theorem to M.)

5.2. Calculus of differential forms In what follows $\mathbb{K}^n := (0,1)^n \subset \mathbb{R}^n$ stands for the open unit cube.

Let $f_{ij} : \mathbb{K}^{n_i} \to \mathbb{R}$, $1 \leq i \leq \ell$, $j = 1, 2$, be continuous functions such that $f_{i1} \leq f_{i2}$ and the closures of sets $\Gamma^{n_i}_{f_{i1},f_{i2}} := \{(x, tf_{i1}(x) + (1-t)f_{i2}(x))\,;\, x \in \mathbb{K}^{n_i},\, t \in [0,1]\}$ are weak k-Markov subsets of $\mathbb{R}^{n_i+1}$ for all $1 \leq i \leq m$. A set of the form $\Gamma^{n_1}_{f_{11},f_{12}} \times \cdots \times \Gamma^{n_\ell}_{f_{\ell 1},f_{\ell 2}} \subset \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_\ell+1}$ $\big(=: \mathbb{R}^n,\ n := \sum_{i=1}^{\ell}(n_i+1)\big)$ will be called a Γ *set*. Each component $\Gamma^{n_i}_{f_{i1},f_{i2}} \subset \mathbb{R}^{n_i+1}$ will be called a *simple* Γ *set*.

By $\mathrm{Mar}^*_{k,\Gamma} \subset \mathrm{Mar}^*_k$ we denote the subclass of sets S such that for each $x \in S$ there exists a chart $(U_\alpha, \varphi_\alpha)$ with $x \in U_\alpha$ and $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ a Γ set.

If M is a C^k manifold, then by $\mathrm{Mar}^*_{k,\Gamma}(M) \subset \mathrm{Mar}^*_{k,\Gamma}$ we denote the family of closed subsets $S \subset M$ such that in the above definition each $(U_\alpha, \varphi_\alpha)$ has the form $U_\alpha := V_\alpha \cap S$ and $\varphi_\alpha := \psi_\alpha|_{U_\alpha}$, where (V_α, ψ_α) is a C^k chart on M.

In particular, every C^k manifold with a boundary, $k \geq 1$, belongs to $\mathrm{Mar}^*_{k,\Gamma}$. However, one can show that the von Koch snowflake curve with the weak ∞-Markov structure induced from $\mathbb{R}^2$ does not belong to $\mathrm{Mar}^*_{\infty,\Gamma}$. The next result shows that this class contains sufficiently many fractal topological manifolds.

THEOREM 5.5. *Let M be a C^k submanifold, $k \geq 1$, of an n-dimensional C^∞ Riemannian manifold X of dimension $m \geq \left\lfloor \frac{n}{2} \right\rfloor \geq 1$ and let $U \subset X$ be an open neighbourhood of M. There is an open neighbourhood $N \subset U$ of M such that*

A. *For each $d \in [m, n]$ there exist a topological submanifold $M_d \subset \mathrm{Mar}^*_{\infty,\Gamma}(N)$ having a base of topology of sets of Hausdorff dimension d and an isotopy $h_{d,t} : M_d \to N$, $t \in [0,1]$, such that $h_{d,0} = \mathrm{id}$, $h_{d,1}(M_d) = M$, each $h_{d,t}(M_d) \in \mathrm{Mar}^*_{\infty,\Gamma}(N)$, $0 \leq t < 1$, and $h_{d,t} : M_d \to h_{d,t}(M_d)$ is a C^k diffeomorphism for all such t. Moreover, M_m is a Lipschitz manifold and $h_{m,1} : M_m \to M$ is locally bi-Lipschitz.*

B. *For each natural number l there exist a Lipschitz submanifold $M_l \in \mathrm{Mar}^*_{l,\Gamma}(N) \setminus \mathrm{Mar}^*_{l+1,\Gamma}(N)$ and an isotopy $h_{l,t} : M_l \to N$, $t \in [0,1]$, such that $h_{l,0} = \mathrm{id}$, $h_{l,1}(M_l) = M$, each $h_{l,t}(M_l) \in \mathrm{Mar}^*_{l,\Gamma}(U) \setminus \mathrm{Mar}^*_{l+1,\Gamma}(N)$, $0 \leq t < 1$ and $h_{l,t} : M_l \to h_{l,t}(M_l)$, is a C^p diffeomorphism, $p := \min\{l, k\}$, for all such t. Moreover, $h_{l,1} : M_l \to M$ is locally bi-Lipschitz.*

For $S_1, S_2 \in \mathrm{Mar}_k^*$, $k \geq 1$, a continuous map $f : S_1 \to S_2$ is said to be *locally Lipschitz* if it is locally Lipschitz in all suitable charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\alpha 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ on S_1 and S_2, respectively (i.e., $\varphi_{\alpha 2} \circ f \circ \varphi_{\beta 1}^{-1} \in \mathrm{Lip}_{\mathrm{loc}}\big((\varphi_{\beta 1} \circ f^{-1})(U_{\alpha 2}), \varphi_{\alpha 2}(U_{\alpha 2})\big)$, where the metric structures of sets in the brackets are induced from those of the corresponding Euclidean spaces). Clearly, the definition does not depend on the choice of charts and each C^k map $S_1 \to S_2$ is locally Lipschitz as well. We say that $f : S_1 \to S_2$ is *locally bi-Lipschitz* if f is a homeomorphism and f, f^{-1} are locally Lipschitz maps. Since the class of locally bi-Lipschitz homeomorphisms preserves Hausdorff dimension and each m-dimensional C^k manifold M, $m \geq 1$, $k \geq 1$, admits a proper C^k embedding into $\mathbb{R}^{2m}$, Theorem 5.5 shows that there is a family of the cardinality of the continuum of locally bi-Lipschitz nonequivalent weak ∞-Markov structures on M.

For $S \in \mathrm{Mar}_k^*$, $k \geq 1$, the *cotangent bundle* $T^*S \to S$ is a bundle dual to the tangent bundle $TS \to S$ and equipped with a weak k-Markov structure induced by that on TS. A *differential form* ω of class C^k on S of degree $\deg \omega = p$ is a C^k section of the bundle $\wedge^p T^*S$. If the rank of TS is n and $x = (x_1, \dots, x_n)$ are coordinates in a chart on S, then ω can be written locally in these coordinates as

$$\omega := \sum_{|I|=p} c_I(x) dx_I,$$

where c_I are of class C^k and for $I = (i_1, \dots, i_s)$, $1 \leq i_\ell \leq n$, $1 \leq \ell \leq s$, we define $|I| := s$ and $dx_I := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_s}$; here $dx_1, \dots, dx_n$ are local coordinates on T^*S in the chart, dual to the coordinates $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ on TS there (pulled back from $T\mathbb{R}^n$ by the map defining this chart).

In what follows we assume that $S \in \mathrm{Mar}_\infty^*$. By $\Omega^p(U)$ we denote the vector space of C^∞ differential forms of degree p on an open set $U \subset S$. The differential $d : \Omega^p(U) \to \Omega^{p+1}(U)$ is defined exactly as in the case of differential forms on C^∞ manifolds (e.g., if $S \in \mathrm{Mar}_\infty^*(\mathbb{R}^n)$ and $\omega = \sum_{|I|=p} c_I \, dx_I \in \Omega^p(S)$, where $x_1, \dots, x_n$ are coordinates on $\mathbb{R}^n$, then $d\omega := \sum_{|I|=p} \big(\sum_{i=1}^n D_S^i c_I \, dx_i\big) \wedge dx_I$).

The following result generalizes the classical Poincaré d-lemma for differential forms on C^∞ manifolds.

Theorem 5.6. *If $S \in \mathrm{Mar}_{\infty,\Gamma}^*$ and $\omega \in \Omega^p(S)$ is d-closed, i.e., $d\omega = 0$, then for each $x \in S$ there exists an open neighbourhood U_x of x and, for $p \geq 1$, a form $\eta_x \in \Omega^{p-1}(U_x)$ such that $d\eta_x = \omega$. If $p = 0$, then ω is locally constant.*

By Ω_S^p we denote the sheaf of germs of C^∞ differential forms of degree p on $S \in \mathrm{Mar}_\infty^*$. Then from Proposition 5.2 we get (see, e.g., [Hi] for basic definitions and results of algebraic topology used below):

Proposition 5.7. *Ω_S^p is a fine sheaf.*

Combining Theorem 5.6 with Proposition 5.7 we obtain straightforwardly

THEOREM 5.8. *Let $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ and $\mathrm{rank}\, TS = n$. Then we have the following resolution of the constant sheaf $\mathbb{R}$ on S by the sheaves of differential forms:*

$$0 \longrightarrow \mathbb{R} \stackrel{d}{\longrightarrow} \Omega^0_S \stackrel{d}{\longrightarrow} \Omega^1_S \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} \Omega^n_S \stackrel{d}{\longrightarrow} 0.$$

In particular, $\check{H}^i(S,\mathbb{R}) = H^i_{dR}(S,\mathbb{R})$, where $\check{H}^i(S,\mathbb{R})$ is the Čech cohomology of sheaf $\mathbb{R}$ on S and

$$H^i_{dR}(S,\mathbb{R}) := \frac{\mathrm{Ker}\big(\Omega^i(S) \stackrel{d}{\longrightarrow} \Omega^{i+1}(S)\big)}{d\big(\Omega^{i-1}(S)\big)}.$$

Since $\check{H}^i(S,\mathbb{R}) = 0$ for $i > \dim_c S$, where $\dim_c$ stands for the covering dimension, Theorem 5.8 yields

COROLLARY 5.9. *For $i > \dim_c S$ each d-closed form in $\Omega^i(S)$ is d-exact.*

The statement of Theorem 5.8 is not valid if $S \notin \mathrm{Mar}^*_{\infty,\Gamma}$ as the following simple example shows.

EXAMPLE 5.10. Let $S \subset \mathrm{Mar}^*_\infty(\mathbb{R})$ be the standard Cantor set in $[0,1]$. Then $\check{H}^0(S,\mathbb{R})$ consists of continuous locally constant functions on S. In particular, each such function assigns finitely many values. Let $[0,1]\setminus S := \cup_{j=1}^\infty (a_j,b_j)$ and φ be a positive C^∞ function on $(0,1)$ having zeros of infinite order at 0 and 1. For a uniformly bounded sequence $\{c_j\} \subset \mathbb{R}_+$ consider a sequence of positive functions $\varphi_j \in C^\infty\big((a_j,b_j)\big)$, $\varphi_j(x) := c_j e^{-(b_j-a_j)}\varphi\big(\frac{x-a_j}{b_j-a_j}\big)$, $j \in \mathbb{N}$. Then the function $\tilde{\varphi}$ equals φ_j on (a_j,b_j), $j \in \mathbb{N}$, and 0 on $S \cup \big(\mathbb{R}\setminus[0,1]\big)$ is C^∞ on $\mathbb{R}$. We set $\psi(x) := \int_{-\infty}^x \tilde{\varphi}(t)\,dt$, $x \in \mathbb{R}$. By the definition $\psi \in C^\infty(\mathbb{R})$ and $d(\psi|_S) = 0$. However, the range of $\psi|_S$ is not finite. This shows that $\check{H}^0(S,\mathbb{R})$ is a proper subset of $H^0_{dR}(S,\mathbb{R})$.

In general, for $S \in \mathrm{Mar}^*_\infty$ as in the case of C^∞ manifolds (using Proposition 5.7) we obtain canonical homomorphisms of groups $h^i_S : \check{H}^i(S,\mathbb{R}) \to H^i_{dR}(S,\mathbb{R})$ such that if $f : S_1 \to S_2$ is a C^∞ map of spaces in Mar^*_∞, then for all i the following diagrams are commutative:

$$\begin{array}{ccc} \check{H}^i(S_2,\mathbb{R}) & \stackrel{h^i_{S_2}}{\longrightarrow} & H^i_{dR}(S_2,\mathbb{R}) \\ f^*_C\Big\downarrow & & f^*_{dR}\Big\downarrow \\ \check{H}^i(S_1,\mathbb{R}) & \stackrel{h^i_{S_1}}{\longrightarrow} & H^i_{dR}(S_1,\mathbb{R}). \end{array} \tag{8}$$

Here f^*_C and f^*_{dR} are homomorphisms induced by f.

As another consequence of Theorem 5.8 we obtain the following interpolation result.

Let $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ be a closed subset of a space $X \in \mathrm{Mar}^*_\infty$ and the weak ∞-Markov structure on S is induced by that on X, i.e., the embedding $S \hookrightarrow X$ induces a surjective morphism $\Omega^0_X \to \Omega^0_S$ of sheaves of germs of C^∞ functions.

Corollary 5.11. *There exists an open neighbourhood $N \subset X$ of S such that for each d-closed form $\omega \in \Omega^p(S)$ there exists a d-closed form $\widetilde{\omega} \in \Omega^p(N)$ such that $\widetilde{\omega}|_S = \omega$.*

By $H^*(S,\mathbb{R})$ we denote the ring of singular cohomology of $S \in \mathrm{Mar}^*_\infty$ endowed with $\smile$ product (see, e.g., [MS, Appendix A] for definitions). Also we consider the ring $H^*_{dR}(S,\mathbb{R})$ endowed with $\wedge$ product. If S is a C^∞ manifold, then the classical de Rham theorem states that these two rings are isomorphic. This result can be extended:

Theorem 5.12. *For $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ there exists an isomorphism of rings $\Phi_S : (H^*_{dR}(S,\mathbb{R}),\wedge) \to (H^*(S,\mathbb{R}),\smile)$ such that if $f : S_1 \to S_2$ is a C^∞ map of spaces in $\mathrm{Mar}^*_{\infty,\Gamma}$, then the following diagram is commutative*

$$
\begin{array}{ccc}
H^*_{dR}(S_2,\mathbb{R}) & \stackrel{\Phi_{S_2}}{\longrightarrow} & H^*(S_2,\mathbb{R}) \\
f^*_{dR}\big\downarrow & & f^*\big\downarrow \\
H^*_{dR}(S_1,\mathbb{R}) & \stackrel{\Phi_{S_1}}{\longrightarrow} & H^*(S_1,\mathbb{R}).
\end{array} \tag{9}
$$

Here f^* is the ring homomorphism induced by f.

Moreover, if S is a C^∞ manifold, then Φ_S coincides with the classical de Rham isomorphism defined by the pairing of differential forms and chains via integration.

Let S_i be elements of $\mathrm{Mar}_{\infty,\Gamma}$, $i = 1,2$. Then $S_1 \times S_2 \in \mathrm{Mar}_{\infty,\Gamma}$ as well. By $\pi_i : S_1 \times S_2 \to S_i$, $i = 1,2$, we denote the natural projections. The map

$$(\omega_1,\omega_2) \mapsto \pi_1^*(\omega_1) \wedge \pi_2^*(\omega_2), \qquad \omega_i \in \Omega^*(S_i), \quad i = 1,2,$$

determines a bilinear map $\wedge : H^*_{dR}(S_1,\mathbb{R}) \otimes H^*_{dR}(S_2,\mathbb{R}) \to H^*_{dR}(S_1 \times S_2,\mathbb{R})$ sending each $\oplus_{p+q=k} H^p_{dR}(S_1,\mathbb{R}) \otimes H^q_{dR}(S_2,\mathbb{R})$ into $H^k_{dR}(S_1 \times S_2,\mathbb{R})$.

As a corollary of Theorem 5.12 we have the following generalization of the classical Künneth formula.

Theorem 5.13. *Suppose S_2 is homotopy equivalent to a CW complex having only finitely many cells in each dimension. Then the map $\wedge : H^*_{dR}(S_1,\mathbb{R}) \otimes H^*_{dR}(S_2,\mathbb{R}) \to H^*_{dR}(S_1 \times S_2,\mathbb{R})$ is an isomorphism.*

In the case where both S_i admit locally finite acyclic open covers, the result was first established by Vladimir Goldshtein by the method analogous to the one described in [BT, §5] (by means of the Mayer-Vietoris sequences which clearly exist in the category of spaces in Mar^*_∞, cf. [BT, §2] for similar arguments). The question whether a general space in $\mathrm{Mar}_{\infty,\Gamma}$ has such a cover is open.

Theorem 5.13 can be applied, e.g., if $S_2 \in \mathrm{Mar}^*_{\infty,\Gamma}$ is a compact topological manifold (in this case it has homotopy type of a finite CW complex, see [KS]).

In a forthcoming paper we present some results, different from those formulated above, on the de Rham cohomology of certain topological manifolds in $\mathrm{Mar}^*_\infty \setminus \mathrm{Mar}^*_{\infty,\Gamma}$ (including von Koch snowflake-type curves) and of Cantor-type sets in $\mathbb{R}^n$ (having similar to the Sierpiński gasket structures).

References

[**BB1**] A. Brudnyi and Yu. Brudnyi, Methods of Geometric Analysis in extension and trace problems, Mathematical Monographs, Springer, Basel, 2011.

[**BB2**] A. Brudnyi and Yu. Brudnyi, Traces of C^k functions to weak Markov subsets of $\mathbb{R}^n$, St. Petersburg Math. J. **23** (2011), 1–19.

[**B**] A. Brudnyi, Differential calculus on topological spaces with weak Markov structure I, to appear, Preprint 2011, 42 pps.

[**BM**] E. Bierstone and P. Milman, C^m norms of finite sets and C^m extension criteria, Duke Math. J. **137** (2007), 118–134.

[**BT**] R. Bott and L. W. Tu, Differential forms in algebraic topology, Springer, New York, 1982.

[**Fa**] K. Falconer, Fractal geometry: mathematical foundations and applications, John Wiley, New York, 1990.

[**F1**] Ch. Fefferman, Whitney's extension problem in certain function spaces, preprint, 2003, 1–78.

[**F2**] Ch. Fefferman, Interpolation and extrapolation of smooth functions by linear operators, Revista Mat. Iberoamericana **21** (2005), 313–348.

[**F3**] Ch. Fefferman, Whitney's extension problems and interpolation of data, Bull. AMS **46** (2009), 207–220.

[**Ha**] G. H. Hardy, Weierstrass's nondifferentiable function, Trans. Amer. Math. Soc. **17** (1916), 301–325.

[**Hes**] M. Hestens, Extension of the range of a differentiable function, Duke Math. J. **8** (1941), 183–192.

[**Hi**] F. Hirzebruch, Topological methods in algebraic geometry, Springer, Berlin, 1966.

[**Hun**] B. R. Hunt, The Hausdorff dimensions of graphs of Weierstrass functions, Proc. Amer. Math. Soc. **126** (1998), 791–800.

[**JW**] A. Jonsson and H. Wallin, Function Spaces on Subsets of $\mathbb{R}^n$, Harwood Academic Publishers, 1984.

[**KMY**] J. L. Kaplan, J. Mallet–Paret, and J. A. Yorke, The Lyapunov dimension of a nowhere differentiable attracting torus, Ergod. Th. Dynam. Sys. **4** (1984), 261–281.

[**KS**] R. C. Kirby and L. C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc. **75**, Number 4 (1969), 742–749.

[**Lu**] G. K. Luli, $C^{m,\omega}$ extension by bounded-depth linear operators, Adv. Math. **224** (2010), 1927–2021.

[**MS**] J. W. Milnor and J. D. Stasheff, Characteristic classes, Princeton, New Jersey, 1974.

[**Mu**] J. R. Munkres, Elementary differential topology, Princeton Univ. Press, 1973.

[**MW**] R. D. Maudlin and S. C. Williams, On the Hausdorff dimension of some graphs, Trans. Amer. Math. Soc. **298** (2) (1986), 793–803.

[**PU**] F. Przytycki and M. Urbanski, On the Hausdorff dimension of some fractal sets, Studia Math. **XCIII** (1989), 155–186.

[**S**] R. S. Strichartz, Analysis on fractals, Notices of the AMS **46** (1999), 1199–1208.

[**Shv**] P. Shvartsman, The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces, Trans. Amer. Math. Soc. **360** (2008), 5529–5550.

[**W1**] H. Whitney, Analytic extension of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. **36** (1934), 63–89.

[**W2**] H. Whitney, Differentiable functions defined in closed sets, I, Trans. Amer. Math. Soc. **36** (1934), 369–387.

[**W3**] H. Whitney, Differentiable manifolds, Ann. Math. **37** (1935), 645–680.

[**W4**] H. Whitney, Geometric integration theory, Princeton University Press, 1957.

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