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# The Goursat problem for the Einstein-Vlasov system: (I) The initial data constraints

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Abstract. We show how to assign, on two intersecting null hypersurfaces, initial data for the Einstein-Vlasov system in harmonic coordinates. As all the components of the metric appear in each component of the stress-energy tensor, the hierarchical method of Rendall cannot apply strictly speaking. To overcome this difficulty, additional assumptions have been imposed to the metric on the initial hypersurfaces. Consequently, the distribution function is constrained to satisfy some integral equations on the initial hypersurfaces.

Résumé. Nous montrons comment construire, sur deux hypersurfaces caractéristiques sécantes, les données initiales pour le problème de Cauchy associé aux équations d'Einstein-Vlasov en jauge harmonique. Comme toutes les composantes de la métrique apparaissent dans chaque composante du tenseur d'impulsion-énergie, la méthode de construction hierarchisée de Rendall ne peut pas s'appliquer stricto sensu. Pour surmonter cette difficulté, une condition supplémentaire est imposée à la métrique sur les hypersurfaces initiales. Par conséquent la fonction de distribution des particules est contrainte à vérifier des équations intégrales sur les hypersurfaces initiales.

## 1. The Einstein-Vlasov (EV) system

*1.1. The complete form of the EV system* The EV system governs the evolution of a collisionless gas (of particles) in General Relativity. The geometric framework is a four dimensional differentiable manifold $\mathcal{M}$, endowed with a hyperbolic metric $\widehat{g}$ of signature $-+++$. The manifold $(\mathcal{M}, \widehat{g})$ is called a space-time. $\mathcal{M}$ is assumed to be orientable and of class $C^\infty$. A particle of rest-mass $m$ is described by a trajectory $s \to (y(s), q(s))$ in the tangent bundle $T\mathcal{M}$ such that $\frac{dy(s)}{ds} = q(s)$ and at each point $y(s)$, the $4-$momentum $q(s)$ of the particle is future oriented and satisfies

$$\widehat{g}_{ij}(y(s))\, q^i(s)\, q^j(s) = -m^2. \qquad \text{(H)}$$

So at each point $y(s)$, $q(s)$ belongs to the future sheet $F_{y(s)}$ of the hyperboloid (H) and its Lorentzian norm is the rest-mass of the particle. Throughout the

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work, commas will be used to denote partial derivatives, e.g., $\widehat{g}_{ij,k} = \frac{\partial \widehat{g}_{ij}}{\partial y^k}$, $\left(y^i\right)$ being local coordinates on $\mathcal{M}$. Roman indices $i, j, ...$ run from 1 to 4 while Greek ones $\alpha, \beta, ...$ run from 3 to 4. The upper-case Roman indices $A, B, ...$ range from 2 to 4. Einstein convention on repeated indices is used i.e., $A_i B^i = \sum_i A_i B^i$. The EV system reads as follows (see [1, 2, 5, 7, 9])

$$\widehat{S}_{ij} \equiv \widehat{R}_{ij} - \frac{1}{2}\widehat{R}\widehat{g}_{ij} = \widehat{T}_{ij}, \quad q^i \frac{\partial f}{\partial y^i} - \widehat{\Gamma}^i_{jk} q^j q^k \frac{\partial f}{\partial q^i} = 0, \tag{1}$$

where $\widehat{g}_{ij}$ are the covariant components of the metric $\widehat{g}$. They constitute the unknowns for the Einstein equations. $\widehat{R}_{ij}$ are the covariant components of the Ricci tensor and $\widehat{R}$ is the scalar curvature of the metric $\widehat{g}$. $\widehat{\Gamma}^k_{ij}$ are the Christoffel symbols of the metric $\widehat{g}$. $f$ is the distribution function (or the particle number density function) which constitutes the unknown for the Vlasov equation. $\widehat{T}_{ij}$ are the covariant components of the stress-energy (or energy-momentum) tensor which is the source of the gravitational field created by the particles. In contravariant components the stress-energy is defined by the following relation (see [1, 2, 5, 7, 9])

$$\widehat{T}^{ij}(y) = -\int_{F_y} f(y, q)\, q^i q^j \frac{|\widehat{g}|^{\frac{1}{2}}}{q_1} d^3 q, \tag{2}$$

where $F_y = \left\{q = \left(q^i\right) \in T_y\mathcal{M} : \widehat{g}_{ij}(y)\, q^i q^j = -m^2,\ 0 < q^1\right\}$, $d^3 q = dq^2 \wedge dq^3 \wedge dq^4$, $|\widehat{g}|$ is the modulus of the determinant of $(\widehat{g}_{ij})$.

*1.2. The reduced EV system* The Einstein equations as they stand are not hyperbolic but in harmonic coordinates they read (see [2, 4])

$$\widehat{R}^h_{ij} = \widehat{T}_{ij},$$

where

$$\widehat{R}^h_{ij} \equiv \widehat{R}_{ij} - \frac{1}{2}\left(\widehat{g}_{ik}\widehat{\Gamma}^k_{,j} + \widehat{g}_{jk}\widehat{\Gamma}^k_{,i}\right) = -\frac{1}{2}\widehat{g}^{km}\widehat{g}_{ij,mk} + Q_{ij}.$$

Here $Q_{ij}$ is a rational function depending on the metric components and their first order derivatives (see [4]),

$$\widehat{\Gamma}^k = \widehat{g}^{ij}\widehat{\Gamma}^k_{ij}. \tag{3}$$

So the reduced EV system in the local coordinates $(y, q)$ reads as follows

$$-\frac{1}{2}\widehat{g}^{km}\widehat{g}_{ij,mk} + Q_{ij} = \widehat{T}_{ij}, \quad q^i \frac{\partial f}{\partial y^i} + Q^i \frac{\partial f}{\partial q^i} = 0, \tag{4}$$

where $Q^i = -\widehat{\Gamma}^i_{jk} q^j q^k$.

*1.3. Appropriate unknowns and variables* As a relativistic speed is bounded, we think that it is convenient to choose on the mass shell, coordinates with bounded domain. Let $y \in \mathbb{U}$ , $\mathbb{U}$ is the domain of a local chart in $\mathcal{M}$. Set $w^A = \frac{q^A}{q^1}$, $A = 2, 3, 4$ and denote by $M_y$ the image in $\mathbb{R}^3$ of $F_y$ by the mapping $\left(q^i\right) \mapsto \left(w^A\right)$. Assume the following hyperbolicity conditions on $(\widehat{g}_{ij})$.

**Assumption** $\widehat{h}$: The metric $(\widehat{g}_{ij})$ is uniformly hyperbolic and the hypersurfaces $y^1 = Cst$ are uniformly spacelike, i.e.,

$$\exists a, b \in (0, \infty) : a^2 \left|\xi\right|^2 \leq \widehat{g}_{AB}\xi^A\xi^B \leq b^2 \left|\xi\right|^2 \tag{5}$$

where $\left|\xi\right|^2 = \sum\limits_{A=2}^{4} \left(\xi^A\right)^2, \quad -\widehat{g}_{11} \geq a^2$ and $-\widehat{g}^{11} \geq a^2$.

PROPOSITION 1. *(i) Under assumption (5), the Vlasov equation reads*

$$q^i \frac{\partial f}{\partial y^i} + Q^A \frac{\partial f}{\partial q^A} = 0. \tag{6}$$

*(ii) Under assumption (5), $M_y$ is a bounded domain in $\mathbb{R}^3$ such that $M_y \subset M$, where $M$ is a fixed compact domain in $\mathbb{R}^3$. The stress-energy tensor (2) is given as follows*

$$\widehat{T}^{ij}(y) = \frac{1}{m^2} \int_{M_y} f(y, w)\, q^i q^j \left(q^1\right)^4 \left|\widehat{g}\right|^{\frac{1}{2}} d^3 w, \tag{7}$$

*where $d^3w = dw^2 \wedge dw^3 \wedge dw^4$, $f(y, w)$ is the expression of $f(y, q)$ in the local coordinates $(y, w)$.*

PROOF. See [2]. □

REMARK 1. (i) In the expression (7) of the stress-energy tensor, we would like to write $f(y, w)\, q^i q^j \left(q^1\right)^4$ as $\varphi(y, w)\, w^{ij}$, where $w^{ij}$ does not depend on $(\widehat{g}_{ij})$. To do so, we proceed to the following change of the unknown distribution function by setting $f(y, w) = \varphi(y, w) \left(q^1\right)^{-6}$, so we must have $w^{ij} = \frac{q^i q^j}{(q^1)^2}$.

(ii) The reduction from $q$ to $w$ corresponds to imposing that the timelike component of the 4−momentum be equal to 1, i.e., to using $y^1$ as time parameter for the particles. This is possible since the trajectories of the particles are timelike and future oriented.

PROPOSITION 2. *Under the change $f(y, w) = \varphi(y, w) \left(q^1\right)^{-6}$, the stress-energy tensor (7) is given as follows*

$$\widehat{T}^{ij}(y) = \frac{1}{m^2} \int_{M_y} \varphi(y, w)\, w^{ij} \left|\widehat{g}\right|^{\frac{1}{2}} d^3 w, \text{ where } w^{ij} = \frac{q^i q^j}{\left(q^1\right)^2}. \tag{8}$$

*The Vlasov equation (6) becomes*

$$q^i \frac{\partial \varphi}{\partial y^i} + \frac{1}{q^1} \left(Q^A - w^A Q^1\right) \frac{\partial \varphi}{\partial w^A} - \frac{6}{q^1} Q^1 \varphi = 0. \tag{9}$$

Proof. See [2]. □

Remark 2. *The expression* (8) *of the stress-energy tensor is not appropriate since the domain* $M_y$ *depends on* $y$ *and makes it difficult to differentiate* $\widehat{T}^{ij}$ *even in the distributional sense. It appears therefore judicious to transform this domain in order to make it independent of* $y$.

Assume the following decomposition of $\widehat{g}$.

**Assumption** $\widehat{h}'$**:** The spatial part of $(\widehat{g}_{AB})$ is decomposed as follows

$$\widehat{g}_{AB}Y^AY^B = \sum_{B=2}^{4}\left(\lambda_A^B Y^A\right)^2, \tag{10}$$

where $\lambda_A^B$ are functions that depends smoothly ($C^\infty$ for instance) on the components $\widehat{g}_{AB}$ of the metric. Set

$$v^C = \left(-\widehat{g}^{11}\right)^{\frac{1}{2}}\lambda_A^C\left[w^A + \widetilde{g}^{1A}\right], \text{ with } \widetilde{g}^{1A} = -\frac{1}{\widehat{g}^{11}}\widehat{g}^{1A}. \tag{11}$$

Proposition 3. *The image of* $M_y$ *by the mapping* $\left(w^A\right) \mapsto \left(v^A\right)$ *is the unit open ball* $B$ *in* $\mathbb{R}^3$. *In the parameters* $\left(v^A\right)$ *defined in* (11), *the energy-momentum tensor* (8) *becomes*

$$\widehat{T}^{ij}(y) = \frac{1}{m^2}\int_B \varphi(y,v)\, v^{ij}\, |\widehat{g}|^{\frac{1}{2}}\left(-\widehat{g}^{11}\right)^{-\frac{3}{2}}|\widetilde{g}|^{-\frac{1}{2}}\, d^3v, \tag{12}$$

*where* $d^3v = dv^2 \wedge dv^3 \wedge dv^4$, $v^{ij} = \frac{q^iq^j}{(q^1)^2}$, $|\widetilde{g}|$ *is the modulus of the determinant of* $(\widehat{g}_{AB})$, $\varphi(y,v)$ *is the expression of* $\varphi(y,w)$ *in the local coordinates* $(y,v)$. *The Vlasov equation* (9) *becomes*

$$\begin{aligned}&\tfrac{\partial\varphi}{\partial y^1} + \left[\left(\lambda_B^A\right)^{-1}v^B\left(-\widehat{g}^{11}\right)^{-\frac{1}{2}} - \widetilde{g}^{1A}\right]\tfrac{\partial\varphi}{\partial y^A}\\ &+\left(q^1\right)^{-2}\left(-\widehat{g}^{11}\right)^{\frac{1}{2}}\lambda_B^A\left(Q^B - w^BQ^1\right)\tfrac{\partial\varphi}{\partial v^A} - 6\left(q^1\right)^{-2}Q^1\varphi = 0.\end{aligned} \tag{13}$$

Proof. See [2]. □

Remark 3. Y. Choquet-Bruhat [2] used assumption (10) and a variant of assumption (5) to treat the ordinary Cauchy problem for the EV system. But in the characteristic case, those assumptions need to be recast through a judicious change of local events variables $\left(y^i\right)$.

Proposition 4. *Let* $\left(y^i\right)$ *be a local coordinates system on* $\mathcal{M}$ *in which the components* $(\widehat{g}_{ij})$ *of the metric satisfy assumption* (5). *Set*

$$x^1 = \frac{1}{2}\left(y^1 + y^2\right), \quad x^2 = \frac{1}{2}\left(y^1 - y^2\right), \quad x^\alpha = y^\alpha, \quad \alpha = 3, 4. \tag{14}$$

*In the coordinates system* $(x,p)$, *the EV system* (1) *reads*

$$S_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}, \quad p^i \frac{\partial f}{\partial x^i} + P^i \frac{\partial f}{\partial p^i} = 0, \tag{15}$$

*where*

$$\begin{aligned} g_{ij}(x) &= \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{g}_{kl}(y), \quad R_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{R}_{kl}(y), \\ R(x) &= g^{ij}(x) R_{ij}(x), \quad T_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \widehat{T}_{kl}(y), \\ p^i &= \frac{\partial x^i}{\partial y^k} q^k, \quad P^i = \frac{\partial x^i}{\partial y^k} Q^k = -\Gamma^i_{jk} p^j p^k, \\ \Gamma^i_{jk} &= \frac{1}{2} g^{ic} \left( g_{ck,j} + g_{cj,k} - g_{jk,c} \right). \end{aligned}$$

PROOF. A direct calculation leads to the desired equations. □

PROPOSITION 5. *The stress-energy tensor* (12) *is given in the local coordinates* $\left(x^i\right)$ *and the local parameters* $\left(v^A\right)$ *as follows*

$$T^{ij}(x) = \frac{1}{2m^2} \int_B \varphi(x,v)\, v^{ij} \left|g\right|^{\frac{1}{2}} \left(-\widehat{g}^{11}\right)^{-\frac{3}{2}} \left|\widetilde{g}\right|^{-\frac{1}{2}} d^3 v, \tag{16}$$

*where* $v^{ij} = \frac{p^i p^j}{(q^1)^2}$, $p^i = \frac{\partial x^i}{\partial y^k} q^k$, $|g|$ *is the modulus of the determinant of* $(g_{ij})$, $|\widetilde{g}|$ *is the modulus of the determinant of* $(\widehat{g}_{AB})$, $\varphi(x,v)$ *is the expression of* $\varphi(y,v)$ *in the local coordinates* $(x,v)$. *The Vlasov equation* (13) *reads*

$$H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0, \tag{17}$$

*where the explicit expressions of* $H^i$, $L^C$ *and* $F$ *are given in [10].*

PROOF. It is straightforward though lengthy. □

REMARK 4. The EV system (15) *in the local coordinates* $(x,v)$ reads as follows

$$R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}, \quad H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0.$$

*The reduced EV system* (4) *in the local coordinates* $(x,v)$ *reads as follows*

$$\widetilde{R}_{ij} = T_{ij}, \quad H^i \frac{\partial \varphi}{\partial x^i} + L^C \frac{\partial \varphi}{\partial v^C} + F\varphi = 0, \tag{18}$$

*where*

$$\widetilde{R}_{ij} \equiv R_{ij} - \frac{1}{2} \left( g_{ki} \Gamma^k_{,j} + g_{kj} \Gamma^k_{,i} \right) = -\frac{1}{2} g^{km} g_{ij,mk} + Q_{ij}. \tag{18a}$$

**2. The constraints problem for the EV system** The task here is the construction of initial data for the reduced EV such that the constraints $\Gamma^k \equiv g^{ij}\Gamma^k_{ij} = 0$, together with other constraints coming from the EV system, are satisfied on $G^1 \cup G^2$, where $G^1$ and $G^2$ are the hypersurfaces in $\mathbb{R}^4$ defined by $x^1 = 0$ and $x^2 = 0$ respectively. It is worth mentioning that the constraints $\Gamma^k = 0$ is the condition that the coordinates be harmonic with respect to the metric. We will need $G^\omega_T = \{x \in G^\omega : 0 \le x^1 + x^2 \le T\}$, $T > 0$, $\omega = 1, 2$. Here the problem is much more difficult than in [4, 8]. This difficulty stems from the appearance of all the components of the metric in any component of the stress-energy tensor. To overcome this toughness, we choose additional free data along the initial hypersurfaces. All the same we try to mimic, as far as possible, the hierarchical method of Rendall [8]. We will see that the supplementary free data force the distribution function to satisfy specific integral equations on the initial hypersurfaces. The treatment of these integral equations is provided in the appendix. The construction will be made in a standard harmonic coordinates system. The existence of such standard harmonic coordinates system has been established by Rendall [8]. For sake of simplicity, only the case of $C^\infty$ data will be discussed. Data of finite differentiability order may be constructed in Sobolev type spaces using energy inequalities and other classical tools as described in [3, 4]. Let us now adapt the method of Rendall [8] to construct $C^\infty$ initial data for the EV system. The assumptions under which the work is achieved are such that only the data $g_{\alpha\beta}$ and $g_{12}$ have to be constructed on $G^1 \cup G^2$, the relations $\Gamma^k = 0$ have to be arranged on $G^1 \cup G^2$, the relations $g_{22,1} = 2g_{12,2}$ and $g_{11,2} = 2g_{12,1}$ have to be established on $G^1$ and $G^2$ respectively. The main result of the work is stated as follows.

THEOREM 1. *Under suitable integral assumptions on the distribution function, there exists initial data for the reduced EV system such that the constraints $\Gamma^k = 0$ are satisfied on $G^1 \cup G^2$ for the corresponding solution of the evolution problem associated to the EV system.*

PROOF. The construction of the data is done fully on $G^1$ and it will be clear that data on $G^2$ are constructed in quite a similar way.

**First step: Construction of $g_{\alpha\beta}$ and $g_{12}$ on $G^1_T$.** Let $T \in (0, \infty)$, $(h_{\alpha\beta}) = \begin{pmatrix} h_{33} & h_{34} \\ h_{34} & h_{44} \end{pmatrix}$ a matrix function with determinant 1 at each point. Set $g_{\alpha\beta} = \Omega h_{\alpha\beta}$, where $\Omega > 0$ is an unknown function called the conformal factor. Assume, for the first free data, as in [4, 8] that

$$g_{22} = g_{23} = g_{24} = 0 \text{ on } G^1_T. \tag{19}$$

Our additional free data assumption is the following

$$g_{11} = g_{13} = g_{14} = 0 \text{ on } G^1_T. \tag{20}$$

Assumptions (19) and (20) mean that the vector fields $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ are null on $G^1$ and orthogonal to $\frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^4}$.

On $G_T^1$, it holds that (see [4, 10])

$$R_{22} = \tfrac{1}{4} g^{12} g^{\alpha\beta} g_{\alpha\beta,2} \left(2g_{12,2} - g_{22,1}\right) + \tfrac{1}{4} g^{\beta\lambda}_{,2} g_{\lambda\beta,2} - \tfrac{1}{2} \left(g^{\alpha\beta} g_{\alpha\beta,2}\right)_{,2}, \tag{21}$$
$$T_{22} = \left(g_{12}\right)^4 K_{22},$$

where $K_{22}(x) = \frac{1}{16m^2} \int_B \varphi(x, v) \left(1 + v^2\right)^2 d^3v$. Assume $g_{22,1} = 2g_{12,2}$ on $G_T^1$. Then, from (19), $\Gamma^1 = 0$ is equivalent to

$$g_{12,2} = \frac{1}{2} g_{12} \frac{\Omega_{,2}}{\Omega}. \tag{22}$$

The equation

$$\frac{1}{4} g^{\alpha\beta}_{,2} g_{\alpha\beta,2} - \frac{1}{2} \left(g^{\alpha\beta} g_{\alpha\beta,2}\right)_{,2} = T_{22}, \tag{23}$$

provides the following non linear second order ODE with the conformal factor $\Omega$ as unknown

$$-\left(\frac{\Omega_{,2}}{\Omega}\right)^2 + \frac{1}{2} h_{\alpha\beta,2} h^{\alpha\beta}_{,2} - 2\left(\frac{\Omega_{,2}}{\Omega}\right)_{,2} = 2K_{22} \left(g_{12}\right)^4. \tag{24}$$

Setting $\Omega = e^V$, then the following system of ODE is derived from (22) and (24) in order to determine the conformal factor together with $g_{12}$

$$2V_{,22} = -\left(V_{,2}\right)^2 - 2K_{22} \left(g_{12}\right)^4 + \frac{1}{2} h_{\alpha\beta,2} h^{\alpha\beta}_{,2}, \quad g_{12,2} = \frac{1}{2} g_{12} V_{,2}. \tag{25}$$

Let $T \in (0, \infty)$. Assume $h_{\alpha\beta}$, $K_{22} \in C^\infty\left(G_T^1\right)$. Take $V_0$, $V_1$, $W_0 \in C^\infty(\Gamma)$, where $\Gamma \equiv G_T^1 \cap G_T^2$. Then there exists $T_1 \in (0, T)$ such that (25) has a unique solution $(V, g_{12}) \in C^\infty\left(G_{T_1}^1\right) \times C^\infty\left(G_{T_1}^1\right)$ satisfying $V = V_0$, $V_{,2} = V_1$, $g_{12} = W_0$ on $\Gamma$. This follows from known local existence and uniqueness results concerning non-linear ODE with $C^\infty$ data in Banach spaces.

**Second step: The condition $g_{22,1} - 2g_{12,2} = 0$ on $G_{T_1}^1$.** It follows from (18a), (21), and (23) that, on $G_{T_1}^1$, the reduced equation $\widetilde{R}_{22} = T_{22}$ is equivalent to the following homogenous ODE with unknown $g_{22,1} - 2g_{12,2}$ (see [4, 8])

$$\left(g^{12}\right)^2 g_{12,2} \left(g_{22,1} - 2g_{12,2}\right) - g^{12} \left(g_{22,1} - 2g_{12,2}\right)_{,2} = 0. \tag{26}$$

Assume $g_{22,1} - 2g_{12,2} = 0$ on $\Gamma$. Then (26) gives $g_{22,1} - 2g_{12,2} = 0$ on $G_{T_1}^1$ and so $\Gamma^1 = 0$ on $G_{T_1}^1$.

**Third step: Relations $\Gamma^\alpha = 0$ on $G_{T_1}^1$.** We seek a combination between $R_{2\alpha}$ and $\Gamma^\alpha$ that will provide an homogenous ODE on $G^1$ with unknown $\Gamma^\alpha$. On $G_T^1$ it holds that (see [4, 10])

$$R_{2\alpha} + \frac{1}{2} g_{\alpha\beta} \Gamma^\beta_{,2} + \left(g^{12} g_{12,2} g_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta,2}\right) \Gamma^\beta = \psi_\alpha, \tag{27}$$

where the functions $\psi_\alpha$ are known on $G^1$. The relation $\Gamma^3 = \Gamma^4 = 0$ on $G^1$ is to be arranged under a suitable choice of the distribution function on $\widehat{G^1} = G^1 \times B$. It is at this level that the Rendall ([8]) method needs to be modified. Assume that the distribution function $\varphi$ is such that

$$T_{2\alpha} = \psi_\alpha \text{ on } G^1_{T_1}. \tag{28}$$

From (18a), (27) and (28), the reduced system $\widetilde{R}_{2\alpha} = T_{2\alpha}$ is equivalent to the following homogenous system of ODE on $G^1_{T_1}$ with unknown $\left(\Gamma^3, \Gamma^4\right)$

$$\begin{aligned} g_{3\beta}\Gamma^\beta_{,2} + \left(g^{12}g_{12,2}g_{3\beta} + \tfrac{1}{2}g_{3\beta,2}\right)\Gamma^\beta = 0, \\ g_{4\beta}\Gamma^\beta_{,2} + \left(g^{12}g_{12,2}g_{4\beta} + \tfrac{1}{2}g_{4\beta,2}\right)\Gamma^\beta = 0. \end{aligned} \tag{29}$$

Assume $\Gamma^\beta = 0$ on $\Gamma$. Then (29) yields $\Gamma^\beta = 0$ on $G^1_{T_1}$.

**Fourth step: Relation $\Gamma^2 = 0$ on $G^1_{T_1}$.** We seek for a combination of $g^{\alpha\beta}R_{\alpha\beta}$, $\Gamma^2$ and $\Gamma^2_{,2}$ that will provide an homogenous ODE on $G^1_{T_1}$ with unknown $\Gamma^2$. On $G^1_{T_1}$ the following combination holds (see [4, 10])

$$g^{\alpha\beta}R_{\alpha\beta} - 2\Gamma^2_{,2} - 2g^{12}g_{12,2}\Gamma^2 = \frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right), \tag{30}$$

where the functions $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are known on $G^1_{T_1}$. In addition to assumption (28) on $G^1_{T_1}$, assume

$$g^{\alpha\beta}T_{\alpha\beta} = \frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right) \text{ on } G^1_{T_1}. \tag{31}$$

From (30) and (31), the reduced system $\widetilde{R}_{\alpha\beta} = T_{\alpha\beta}$ provides the following homogenous ODE on $G^1_{T_1}$ with unknown $\Gamma^2$

$$-2\Gamma^2_{,2} - 2g^{12}g_{12,2}\Gamma^2 = 0. \tag{32}$$

Assume $\Gamma^2 = 0$ on $\Gamma$. Then it follows from (32) that $\Gamma^2 = 0$ on $G^1_{T_1}$. □

REMARK 5. (*i*) It is easy to see that assumption (28) is a constraint integral system for the distribution function $\varphi$, that is written explicitly as follows

$$\begin{aligned} \textstyle\int_B \varphi(x,v)\left(1+v^2\right)v^3 d^3v = \tfrac{128m^4}{(g_{12})^7\Omega}\left(\psi_4 B_3 - \psi_3 B_4\right), \quad x \in G^1_{T_1}, \\ \textstyle\int_B \varphi(x,v)\left(1+v^2\right)v^4 d^3v = \tfrac{128m^4}{(g_{12})^7\Omega}\left(\psi_3 A_4 - \psi_4 A_3\right), \quad x \in G^1_{T_1}. \end{aligned} \tag{IS}$$

Similarly, assumption (31) is a supplementary integral equation for the distribution function $\varphi$, which is written explicitly as follows

(IE)

$$\begin{aligned} &\tfrac{(-g_{12})^3}{8m^2}\textstyle\int_B \varphi(x,v)\left(v^3\right)^2 d^3v + \tfrac{(-g_{12})^3 h_{34}}{4m^2}\left(1 - (\Omega h_{33})^{\frac{1}{2}}\right)\textstyle\int_B \varphi(x,v)\, v^3 v^4 d^3v \\ &+ \tfrac{(-g_{12})^3}{8m^2}\left[(h_{34})^2(\Omega h_{33})^{\frac{1}{2}}\left((\Omega h_{33})^{\frac{1}{2}} - 2\right) + h_{44}h_{33}\right]\textstyle\int_B \varphi(x,v)\left(v^4\right)^2 d^3v \\ &= \tfrac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta} + M_{\alpha\beta}\right), \quad x \in G^1_{T_1}. \end{aligned}$$

(*ii*) The treatment of the constraints integral equations $(IS)$ and $(IE)$ is done in the appendix.

(*iii*) It would be interesting to see whether the constraints integral equations can be avoided. One way of doing this is to work in temporal gauge and null moving frame (see [6]). But in the harmonic gauge case the issue may not be evident. Nevertheless we think that one could use an orthonormal frame in order to avoid the metric appearing (in an involved way) in the energy-momentum tensor (see [7]).

## Appendix: Treatment of the integral constraints equations

*A.1. The integral system* $(IS)$ The integral system $(IS)$ is written as follows

$$\langle \varphi_x, f\rangle = h_1(x), \quad \langle \varphi_x, g\rangle = h_2(x). \tag{A.1}$$

where $\langle , \rangle$ denotes the scalar product in $L^2(B)$ and

$$\begin{array}{l} f(v) = \left(1+v^2\right)v^3, \quad g(v) = \left(1+v^2\right)v^4, \\ h_1 = \frac{128m^4}{(-g_{12})^7\Omega}\left(\psi_3 B_4 - \psi_4 B_3\right), \quad h_2 = \frac{128m^4}{(-g_{12})^7\Omega}\left(\psi_4 A_3 - \psi_3 A_4\right). \end{array} \tag{A.2}$$

As the distribution function must be non-negative, let us seek $\varphi$ of the form

$$\varphi_x(v) \equiv \varphi(x,v) = \left[a(x) f(v) + b(x) g(v) + c(x)\right]^2, \tag{A.3}$$

where $a(x)$, $b(x)$ and $c(x)$ are unknown functions defined on $G^1$. Expanding (A.3), we have

$$\varphi = a^2 f^2 + b^2 g^2 + 2abfg + 2acf + 2bcg + c^2, \tag{A.4}$$

where variables have been dropped for simplicity. Let $\varphi$ be like in (A.4). It holds that

(A.5)

$$\begin{array}{l} \langle \varphi_x, f\rangle = a^2\langle f^2, f\rangle + b^2\langle g^2, f\rangle + 2ab\langle fg, f\rangle + 2ac\langle f, f\rangle + 2bc\langle g, f\rangle + c^2\langle 1, f\rangle, \\ \langle \varphi_x, g\rangle = a^2\langle f^2, g\rangle + b^2\langle g^2, g\rangle + 2ab\langle fg, g\rangle + 2ac\langle f, g\rangle + 2bc\langle g, g\rangle + c^2\langle 1, g\rangle. \end{array}$$

Spherical coordinates will be used to calculate each of the following quantities that are needed.

(A.6)

$$\begin{array}{l} \langle f^2, f\rangle = \int_B \left[\left(1+v^2\right)v^3\right]^3 d^3v, \quad \langle g^2, f\rangle = \int_B \left[\left(1+v^2\right)v^4\right]^2 \left(1+v^2\right)v^3 d^3v, \\ \langle f, f\rangle = \int_B \left[\left(1+v^2\right)v^3\right]^2 d^3v, \quad \langle fg, f\rangle = \int_B \left[\left(1+v^2\right)v^3\right]^2 \left(1+v^2\right)v^4 d^3v, \\ \langle g, f\rangle = \int_B \left(1+v^2\right)^2 v^3 v^4 d^3v, \quad \langle 1, f\rangle = \int_B \left(1+v^2\right)v^3 d^3v, \\ \langle 1, g\rangle = \int_B \left(1+v^2\right)v^4 d^3v, \quad \langle g^2, g\rangle = \int_B \left[\left(1+v^2\right)v^4\right]^3 d^3v, \\ \langle g, g\rangle = \int_B \left[\left(1+v^2\right)v^4\right]^2 d^3v. \end{array}$$

As $B$ is the open unit ball in $\mathbb{R}^3$, we set

$$v^2 = r\cos\theta\cos\lambda, \quad v^3 = r\cos\theta\sin\lambda, \quad v^4 = r\sin\theta, \tag{A.7}$$

with

$$0 \le r < 1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \lambda \le 2\pi. \tag{A.8}$$

Define a domain $P$ in $\mathbb{R}^3$ as follows

$$P = \left\{ (r, \theta, \lambda) \in \mathbb{R}^3 : 0 \le r < 1, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le \lambda \le 2\pi \right\}. \tag{A.9}$$

Using the change of variables (A.7), we gain

$$\begin{array}{l} \langle f^2, f \rangle = \int_P f^3 r^2 \cos\theta dP, \quad \langle g^2, f \rangle = \int_P g^2 f r^2 \cos\theta dP \\ \langle fg, f \rangle = \int_P f^2 g r^2 \cos\theta dP, \quad \langle f, f \rangle = \int_P f^2 r^2 \cos\theta dP, \\ \langle g, f \rangle = \int_P g f r^2 \cos\theta dP, \quad \langle 1, f \rangle = \int_P f r^2 \cos\theta dP, \\ \langle 1, g \rangle = \int_P g r^2 \cos\theta dP, \quad \langle g^2, g \rangle = \int_P g^3 r^2 \cos\theta dP, \\ \langle g, g \rangle = \int_P g^2 r^2 \cos\theta dP, \end{array} \tag{A.10}$$

where

$$dP = dr d\theta d\lambda.$$

After expansion and reduction we integrate the above quantities over $P$ to obtain

$$\begin{array}{l} \langle f^2, f \rangle = \langle g^2, f \rangle = \langle f^2, g \rangle = \langle g, f \rangle = \langle 1, f \rangle = \langle 1, g \rangle = \langle g^2, g \rangle = 0, \\ \langle f, f \rangle = \langle g, g \rangle = \frac{32\pi}{105}. \end{array} \tag{A.11}$$

From (A.5) and (A.11) it holds that

$$\langle \varphi_x, f \rangle = \frac{64\pi a(x) c(x)}{105}, \quad \langle \varphi_x, g \rangle = \frac{64\pi b(x) c(x)}{105}. \tag{A.12}$$

Thus, $\varphi$ solves (A.1) if and only if

$$\frac{64\pi a(x) c(x)}{105} = h_1(x), \quad \frac{64\pi b(x) c(x)}{105} = h_2(x). \tag{A.13}$$

In summary $\varphi$ is given by

$$\begin{array}{l} \varphi_x(v) \equiv \varphi(x, v) = \left[ a(x) \left(1 + v^2\right) v^3 + b(x) \left(1 + v^2\right) v^4 + c(x) \right]^2, \\ \qquad x \in G^1, \ v = \left(v^2, v^3, v^4\right) \in B, \end{array} \tag{A.14}$$

where

$$a(x) c(x) = \frac{105 h_1(x)}{64\pi}, \quad b(x) c(x) = \frac{105 h_2(x)}{64\pi}. \tag{A.15}$$

*A.2. The integral equation* $(IE)$ The integral equation $(IE)$ is written as follows

$$E\langle \varphi_x, e\rangle + I\langle \varphi_x, i\rangle + S\langle \varphi_x, s\rangle = C, \tag{A.16}$$

where

$$\begin{array}{l} E\left(x\right)=\frac{\left(-g_{12}\right)^{3}}{8m^{2}}, \quad e\left(v\right)=\left(v^{3}\right)^{2}, \quad I\left(x\right)=\frac{\left(-g_{12}\right)^{3}h_{34}}{4m^{2}}\left(1-\left(\Omega h_{33}\right)^{\frac{1}{2}}\right), \\ \qquad i\left(v\right)=v^{3}v^{4}, \quad s\left(v\right)=\left(v^{4}\right)^{2}, \\ S\left(x\right)=\frac{\left(-g_{12}\right)^{3}}{8m^{2}}\left[\left(h_{34}\right)^{2}\left(\Omega h_{33}\right)^{\frac{1}{2}}\left(\left(\Omega h_{33}\right)^{\frac{1}{2}}-2\right)+h_{44}h_{33}\right], \\ C\left(x\right)=\frac{1}{4}g^{\alpha\beta}\left(N_{\alpha\beta}+M_{\alpha\beta}\right). \end{array} \tag{A.17}$$

Using the expression of $\varphi$ given in (A.4) we gain

$$\langle \varphi_x, e\rangle = a^2\langle f^2, e\rangle + b^2\langle g^2, e\rangle + 2ab\langle fg, e\rangle + 2ac\langle f, e\rangle + 2bc\langle g, e\rangle + c^2\langle 1, e\rangle. \tag{A.18}$$

As in the preceding paragraph, the above quantities are found to be

$$\langle f^2, e\rangle = \tfrac{8\pi}{63}, \quad \langle g^2, e\rangle = \tfrac{8\pi}{189}, \quad \langle 1, e\rangle = \tfrac{4\pi}{15}, \quad \langle fg, e\rangle = \langle f, e\rangle = \langle g, e\rangle = 0. \tag{A.19}$$

(A.18) and (A.19) imply

$$\langle \varphi_x, e\rangle = \frac{8\pi}{63}a^2 + \frac{8\pi}{189}b^2 + \frac{4\pi}{15}c^2. \tag{A.20}$$

Similarly it holds that

$$\langle \varphi_x, i\rangle = a^2\langle f^2, i\rangle + b^2\langle g^2, i\rangle + 2ab\langle fg, i\rangle + 2ac\langle f, i\rangle + 2bc\langle g, i\rangle + c^2\langle 1, i\rangle. \tag{A.21}$$

Straightforward calculations as above give

$$\langle f^2, i\rangle = \langle g^2, i\rangle = \langle f, i\rangle = \langle g, i\rangle = \langle 1, i\rangle = 0, \quad \langle fg, i\rangle = \frac{8\pi}{189}. \tag{A.22}$$

(A.21) and (A.22) give

$$\langle \varphi_x, i\rangle = \frac{16\pi ab}{189}. \tag{A.23}$$

We are then left with calculating

$$\langle \varphi_x, s\rangle = a^2\langle f^2, s\rangle + b^2\langle g^2, s\rangle + 2ab\langle fg, s\rangle + 2ac\langle f, s\rangle + 2bc\langle g, s\rangle + c^2\langle 1, s\rangle. \tag{A.24}$$

By proceeding as above we get

$$\langle fg, s\rangle = \langle f, s\rangle = \langle g, s\rangle = 0, \quad \langle f^2, s\rangle = \frac{8\pi}{189}, \quad \langle g^2, s\rangle = \frac{8\pi}{63}, \quad \langle 1, s\rangle = \frac{4\pi}{15}. \tag{A.25}$$

(A.24) and (A.25) yield

$$\left\langle \varphi_x, s \right\rangle = \frac{8\pi}{189}a^2 + \frac{8\pi}{63}b^2 + \frac{4\pi}{15}c^2. \tag{A.26}$$

From (A.16), (A.20), (A.23) and (A.26), we see that the integral equation $(IE)$ is equivalent to

$$10\left(3E+S\right)a^2 + 10\left(3S+E\right)b^2 + 20Iab + 63\left(E+S\right)c^2 = \frac{945}{4\pi}C. \tag{A.27}$$

In view of (A.15), multiplying (A.27) by $c^2$ and rearranging , we gain

$$\begin{aligned} &110\,250\left(3E\left(x\right)+S\left(x\right)\right)\left(h_1\left(x\right)\right)^2 + 110\,250\left(3S\left(x\right)+E\left(x\right)\right)\left(h_2\left(x\right)\right)^2 \\ &+220\,500h_1\left(x\right)h_2\left(x\right)I\left(x\right) + 258\,048\pi^2\left(E\left(x\right)+S\left(x\right)\right)\left[c\left(x\right)\right]^4 \\ &= 967\,680\pi C\left(x\right)\left[c\left(x\right)\right]^2. \end{aligned} \tag{A.28}$$

(A.28) is an algebraic equation that can be solved under suitable assumptions to find $c\left(x\right)$. Doing so we deduce $a\left(x\right)$ and $b\left(x\right)$ thanks to (A.15). Finally the distribution function $\varphi$ is obtained on $\widehat{G^1}$ and has the form (A.14) .

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