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On the Diophantine Equation $x^3 + by + 1 - xyz = 0$

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Abstract. In this paper, we shall prove that all positive integral solutions (x, y, z) of the diophantine equation $x^3 + by + 1 - xyz = 0$ satisfy $x \le b\left((2b^3+b)^3+1\right)+1$, $y \le (2b^3+b)^3+1$, and $z \le \left(b\left((2b^3+b)^3+1\right)+1\right)^2 + 2b^3 + b$ for a given positive integer b. As an application of this result, we investigate the divisors of the sequence $\{n^3+1\}$ in residue classes. More precisely, we study the following sums:

$$\sum_{b \le X} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 \quad \text{and} \quad \sum_{n \le X} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1$$

for a given positive real number X and a positive integer b.

1. Introduction

Consider the diophantine equation

$$x^3 + by + 1 - xyz = 0, \tag{1}$$

where b is a fixed positive integer and x, y and z are unknown positive integers. This equation has been studied by many authors including Mohanty [4], Utz [8], Mohanty-Ramasamy [5] and [6], Luca-Togbe [3], Subburam [7], etc. In 1984, Mohanty and Ramasamy in [5] suggested the following conjecture.

Conjecture 1. *The number (denoted by $N(b)$) of all positive integral solutions (x, y, z) of the diophantine equation $x^3 + by + 1 - xyz = 0$ is less than or equal to $8b + 15$, for any positive integer b.*

Recently, Subburam [7] proved the above conjecture for all large enough b. More precisely, he proved:

Theorem A. (Subburam [7]) *We have*

$$N(b) \le 6b + t(b),$$

where $t(b) = o(b)$ as $b \to \infty$ and $t(b) < b$ for all $b \ge c$, for some computable constant c depending on b.

In this paper, we prove the following theorems.

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THEOREM 1. *Any positive integral solution* (x, y, z) *of* (1) *satisfies*

(1) $x \leq b\left((2b^3+b)^3+1\right)+1$;
(2) $y \leq (2b^3+b)^3+1$;
(3) $z \leq \left(b\left((2b^3+b)^3+1\right)+1\right)^2+2b^3+b$.

In the following theorem, we give a closed formula for $N(b)$.

THEOREM 2. *Let* b *be a given positive integer. Then*

$$N(b) = \sum_{n=1}^{\infty} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1.$$

In view of Theorem A, we have the following Corollary.

COROLLARY 1. *Let* b *be a given positive integer. Then*

$$\sum_{n=1}^{\infty} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 \leq 6b + t(b),$$

where $t(b)$ *is the integer stated in Theorem A.*

In 1984, H. W. Lenstra [2] considered the problem of divisors in residue classes as follows. For every real number $\alpha > 1/4$, there exists a constant $\kappa(\alpha)$ with the following property. If r, s and N are integers such that $0 \leq r < s < N$, $s > N^\alpha$ and $(r, s) = 1$, then he proved that there are at most $\kappa(\alpha)$ positive divisors of N which are congruent to r modulo s. Also, in the same paper, he showed that if $\alpha > 1/3$, then $\kappa(\alpha) = 11$. In 2007, Coppersmith et al. [1] showed that if $\alpha > 0.331$, then $\kappa(\alpha) = 32$. This result will imply that if $n \geq 4$ and b are any positive integers, then

$$\sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 \leq 32.$$

In this paper, as an application of Theorems 1 and 2, we obtain a refinement in the case where $N = n^3 + 1$ and $s = n$ in the above problem. More precisely, we prove:

THEOREM 3. *Let* b *be a given positive integer. Then for every integer* $n > b\left((2b^3+b)^3+1\right)+1$, *we have*

$$\sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 = 0.$$

In other words, if $n > b\left((2b^3+b)^3+1\right)+1$, *then no divisor of* n^3+1 *is congruent to* $-b$ *modulo* n.

If $b \le X$ and $n > X\left((2X^3 + X)^3 + 1\right) + 1$, then we have

$$n > b\left((2b^3 + b)^3 + 1\right) + 1.$$

Hence, by Theorem 3, we have the following corollary.

COROLLARY 2. *Let X be any real number. Then for every integer n satisfying*

$$n > X\left((2X^3 + X)^3 + 1\right) + 1,$$

we have

$$\sum_{b \le X} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 = 0.$$

2. Proof of Theorem 1 To prove this theorem, we need the following lemma.

LEMMA 1 (Subburam [2]). *Let (x, y, z) be a positive integral solution of (1). Then there exists a positive integral solution $(l(x), y, l(z))$ of (1) satisfying*

(1) $xl(x) = by + 1$.
(2) *If $l(x) \ge x$, then $l(z) > b$ and if $x \ge l(x)$, then $z > b$.*
(3) *If $x \ge 3$, and $l(x) \ge x + 2$, then $z \le b$ and if $l(x) \ge 3$, $x \ge l(x) + 2$, then $l(z) \le b$.*

Let (x, y, z) be any positive integral solution of (1).

Case 1. $z \le b$.

Since $x^3 + 1 = (xz - b)y$, we have $(xz - b) \mid (x^3 + 1)$. We see that $(xz - b) \mid (z^3 + b^3)$. This is because

$$z^3(x^3 + 1) = (xz - b)(z^2x^2 + xbz + b^2) + (z^3 + b^3).$$

Therefore

$$(xz - b) \le (z^3 + b^3) \le 2b^3.$$

Hence, as $z \ge 1$, we get

$$x \le 2b^3 + b.$$

Since $y \le x^3 + 1$, we get $y \le (2b^3 + b)^3 + 1$, and so

$$x \le 2b^3 + b, y \le (2b^3 + b)^3 + 1 \text{ and } z \le b.$$

Case 2. $z > b$.

By Lemma 1, there exists a positive integral solution $(l(x), y, l(z))$ of (1) satisfying $xl(x) = by + 1$.

If $x \geq 3$ and $l(x) \geq x+2$, then, by Lemma 1, we get $z \leq b$, which is a contradiction. Hence either $x < 3$ or $l(x) < x+2$.

If $l(x) \geq 3$ and $x \geq l(x)+2$, then, by Lemma 1, we have $l(z) \leq b$. Therefore, by Case 1, we get

$$l(x) \leq 2b^3 + b \text{ and } y \leq (2b^3+b)^3 + 1.$$

Since $xl(x) = by + 1$,

$$x \leq by + 1 \leq b((2b^3+b)^3+1)+1.$$

Since $x^2 + l(x) = yz$, we get

$$z \leq x^2 + l(x) \leq \big(b((2b^3+b)^3+1)+1\big)^2 + 2b^3 + b.$$

If $x = 1$ or $l(x) = 1$, then all the possible integral solutions (x, y, z) of (1) are $(1, 1, b+2), (1, 2, b+1), (b+1, 1, b^2+2b+2), (2b+1, 2, 2b^2+2b+1)$.

If $x = 2$ or $l(x) = 2$, then all the possible integral solutions (x, y, z) of (1) are $(2, 1, (b+9)/2)$, $(2, 3, (b+3)/2)$, $(2, 9, (b+1)/2)$, $\big((b+1)/2, 1, (b^2+2b+9)/4\big)$, $\big((3b+1)/2, 3, (3b^2+2b+3)/4\big)$, $\big((9b+1)/2, 9, (9b^2+2b+1)/4\big)$.

For the case $x = l(x)$, all the positive integral solutions (x, y, z) of (1) satisfy

$$x \in \left\{x \in \mathbb{N} : (x-1) \mid b \text{ and } b \mid (x^2-1)\right\}.$$

When $x = l(x)+1$ or $l(x) = x+1$, then the possible solutions (x, y, z) of (1) are

$$\Big((-1+\sqrt{4b+5})/2, 1, b+2\Big) \text{ and } \Big((1+\sqrt{4b+5})/2, 1, b+1+\sqrt{4b+5}\Big).$$

Thus, in all the cases, we observe that the positive integral solutions (x, y, z) of (1) satisfy

$$x \leq b((2b^3+b)^3+1)+1, y \leq (2b^3+b)^3+1$$

and

$$z \leq \big(b((2b^3+b)^3+1)+1\big)^2 + 2b^3 + b.$$

3. Proof of Theorem 2

Let n be any positive integer. Let d be a positive divisor of n^3+1 such that $d \equiv -b \pmod{n}$. Then there exists a positive integer m such that $d = mn - b$. Since $mn - b \mid n^3+1$, there is a positive integer y such that $n^3 + 1 = (mn-b)y$ which in turn satisfies $n^3 + by + 1 = myn$. That is, for a positive divisor d of n^3+1 with $d \equiv -b \pmod{n}$, we get a positive integral solution (n, y, m) of (1). Indeed, for any two distinct positive divisors d_1 and

d_2, $d_1 \equiv -b \pmod{n}$ and $d_2 \equiv -b \pmod{n}$, of n^3+1, we get distinct positive integral solutions of (1). Therefore, we get,

$$\sum_{n=1}^{\infty} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n}}} 1 \leq N(b).$$

For the other inequality, let (n, y, z) be a positive integral solution of (1). Then we see that $(nz - b)$ divides $n^3 + 1$ and $nz - b$ is positive as y and $n^3 + 1$ are positive. By letting $d = nz - b$, we get a positive divisor of n^3+1 which is $\equiv -b \pmod{n}$. Thus, we get

$$N(b) \leq \sum_{n=1}^{\infty} \sum_{\substack{d \mid n^3+1 \\ d \equiv -b \pmod{n})}} 1.$$

These inequalities prove the theorem.

4. **Proof of Theorem 3** Suppose that there is a positive divisor d of n^3+1 such that $d \equiv -b \pmod{n}$. Then there exists two positive integers y and z such that $n^3 + by + 1 = nyz$. Thus (n, y, z) is a positive integral solution of (1). Therefore, by Theorem 1, we get $n \leq b\left((2b^3+b)^3+1\right)+1$, which is a contradiction to the assumption that $n > b\left((2b^3+b)^3+1\right)+1$. This proves the theorem.

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