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Uniqueness of the Index Map in Banach Algebra K-Theory

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Abstract. It is shown that the index map in Banach algebra K-theory, as a natural map from the K_1-group of a quotient of a Banach algebra to the K_0-group of the corresponding ideal, is unique (up to an integral multiple).

Résumé. Il est démontré que l'application index dans la K-théorie des algèbres de Banach est unique, dans un sens très naturel.

1. Recall (see, e.g., [5]) that the index map in Banach algebra K-theory is a map from the K_1-group of the quotient of a Banach algebra by a closed two-sided ideal to the K_0-group of the ideal, which is natural with respect to maps between extensions of Banach algebras, i.e., commutative diagrams (of contractions)

$$\begin{array}{ccccccccc} 0 & \to & J & \to & A & \to & A/J & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J' & \to & A' & \to & A'/J' & \to & 0 \end{array}$$

in which the rows are short exact sequences. Thus, the maps

$$(A, J) \mapsto K_1(A/J) \text{ and } (A, J) \mapsto K_0(J)$$

are functors and the index map is a natural transformation between them.

Theorem *The index map is unique up to an integral multiple, as a natural transformation as specified above.*

2. Note that Theorem 1 does not refer to any of the properties of the index map (e.g., the long exact sequence, or even the definition!) other than naturality—and the property that it is not a non-trivial integral multiple of any other natural map. In fact, the theorem follows immediately, using these two very basic properties of the index map (together, of course, with the fact that it actually exists!), from the following statement which doesn't even mention the index map—of course the proof uses it!

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Theorem *The additive group of natural transformations between the two functors, from the category of Banach algebra extensions*

$$0 \to J \to A \to A/J \to 0$$

to the category of abelian groups,

$$(A, J) \mapsto \mathrm{K}_1(A/J) \text{ and } (A, J) \mapsto \mathrm{K}_0(J)$$

is singly generated and torsion free (i.e., either $\mathbb{Z}$ *or 0).*

3. *Proof of Theorem 2.* Of course, Theorem 2 follows immediately from Theorem 1, even with the stronger conclusion that the group of natural transformations in question is $\mathbb{Z}$ (not 0)—but it is of interest to observe that Theorem 2, as stated, might conceivably be proved without using the index map (and the existence of this non-zero map then just exclude the alternative 0). Somewhat surprisingly, perhaps, the index map is in fact used in the proof.

To effect the proof, in the general Banach algebra case (see Section 5, below, for a more direct proof in the case of C*-algebras), it is necessary to use Bott periodicity, and to pass by suspension to the analogous (equivalent!) question for natural transformations from

$$\mathrm{K}_0(A/J) \text{ to } \mathrm{K}_1(J).$$

As a simple source of reference extensions, consider the universal Banach algebra generated by an element of norm one. Note that, whereas otherwise it would be possible to consider arbitrary continuous homomorphisms, here it is necessary to restrict to contractions. Let us denote this Banach algebra by U, and the canonical generator by u.

Let $0 \to J \to A \to A/J \to 0$ be an extension of Banach algebras, and let e be an idempotent in A/J (after replacing A by a matrix algebra over the unitization of A, which is what must be considered in the first instance).

Choose $a \in A$ with image (the idempotent) e in A/J. We may suppose that e is non-zero, and so also a. Note that U is also the universal Banach algebra (again recall that we are in the category of Banach algebras with contractions as maps) generated by an element of norm equal to $\|a\|$, namely, the element $\|a\|u$, and so there exists a (unique) map (a contraction) from U into A taking $\|a\|u$ into a. There is then a (unique) map (a contraction) from U into A/J taking $\|a\|u$ into the idempotent e. Thus, we have an extension of Banach algebras (with contractions)

$$0 \to I \to U \to \mathbb{C} \to 0,$$

together with a map of this into the given extension:

$$\begin{array}{ccccccccc} 0 & \to & I & \to & U & \to & \mathbb{C} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J & \to & A & \to & A/J & \to & 0 \end{array}$$

Since U is the universal Banach algebra generated by a contraction, it is homotopy equivalent to 0, and so in particular has trivial K-groups, and so by the six-term exact sequence of Bott periodicity the odd index map (or Bott periodicity map) $\mathbb{Z} = \mathrm{K}_0\mathbb{C} \to \mathrm{K}_1 I$ is an isomorphism, i.e., $\mathrm{K}_1 I$ is isomorphic to the group $\mathbb{Z}$, and so any map at all from $\mathrm{K}_0\mathbb{C}$ to $\mathrm{K}_1 I$ is an integral multiple of the index map. By naturality, it follows that a given natural transformation from $\mathrm{K}_0(A/J)$ to $\mathrm{K}_1 J$ is an integral multiple of the index map at least on the single projection e.

It remains to show that the integral multiple is always the same, for any idempotent e belonging to any quotient A/J. This is seen, for $e_1 \in A_1/J_1$ and $e_2 \in A_2/J_2$, by considering

$$e_1 \oplus e_2 \;\in\; A_1/J_1 \oplus A_2/J_2 = (A_1 \oplus A_2)/(J_1 \oplus J_2).$$

4. *Proof of Theorem 1.* Theorem 1 follows immediately from Theorem 2, on noticing that since the Fredholm index of the unilateral shift is ± 1 (depending on the convention followed), the index map must generate the group of natural transformations in question), which by Theorem 2 must be isomorphic to $\mathbb{Z}$ (since it is non-zero—as the index map is non-zero).

5. Remark If one considers the category of C*-algebras, with C*-algebra maps, then the analogous result, in which one considers only C*-algebra maps between extensions, still holds, but does not quite follow as a consequence, as the reference extension used in the proof in the Banach algebra setting is not an extension of C*-algebras. However, for a natural transformation $\mathrm{K}_0(A/J) \to \mathrm{K}_1 J$, the reference extension of C*-algebras

$$0 \to C_0(]0,1[) \to C_0(]0,1]) \to \mathbb{C} \to 0$$

may be used to determine it, as $C_0(]0,1])$ is the universal C*-algebra generated by a positive element of norm one, and any projection in a C*-algebra quotient always lifts to a positive element of norm at most one. (Note that this extension is all that is needed whereas in the Banach algebra setting many different ideals of U of codimension one—all of them!—must be used.)

In fact we don't need Bott peridicity to prove (essential) uniqueness of the index map, as a natural transformation $\mathrm{K}_1(A/J) \to \mathrm{K}_0 J$, in the C*-algebra setting, since we may use in the same way as for Banach algebras various extensions associated with the universal C*-algebra generated by a contraction (which should not be confused with U, which is not a C*-algebra!). Namely if (C, c) denotes this universal object then, given any unitary element u of a quotient A/J of a unital C*-algebra A (it is enough by naturality to consider the case that A is unital), C may be mapped into A in such a way that c maps into a given lifting of u to an element of A of norm one (assuming that A is non-zero), the combined mapping giving rise to a C*-algebra extension

$$0 \to I \to C \to D = C^*(u) \to 0.$$

There are two cases. (It is at this point that the general Banach algebra setting becomes intractable.) If D is not isomorphic to $\mathrm{C}(\mathbb{T})$, then $\mathrm{K}_1(D) = 0$, and so the given natural transformation $\mathrm{K}_1(A/J) \to \mathrm{K}_0 J$ is zero on the class of u. If D is isomorphic to $\mathrm{C}(\mathbb{T})$, then, again as $\mathrm{K}_* C = 0$ (C is contractible), the index map $\mathrm{K}_1 D \to \mathrm{K}_0 I$ is an isomorphism, in which case, since $\mathrm{K}_1 D = \mathbb{Z}$ and so $\mathrm{K}_0 I$ is isomorphic to $\mathbb{Z}$, any map whatsoever from $\mathrm{K}_1 D$ to $\mathrm{K}_0 I$ is an integral multiple of the index map. Hence a given natural transformation $\mathrm{K}_1(A/J) \to \mathrm{K}_0 J$ must agree on u with an integral multiple of the index map, and by the same direct sum technique as in the proof for the odd index map in the setting of Banach algebras, this multiple is seen to be universal.

6. Remark The related question in the setting of Fréchet algebras (see [4]) on the one hand, or the setting of real Banach algebras (or C*-algebras), on the other hand (see [1]), would seem to be of interest. (For instance, by analogy with the preceding remark, one might consider maps from the universal real C*-algebra generated by a contraction to the universal real C*-algebra generated by a unitary.)

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