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ISOMETRIC COMPOSITION OPERATORS ON THE BLOCH-TYPE SPACES

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ABSTRACT. We show that the only isometric composition operators on the Bloch-type spaces B^α for $0 < \alpha < 1$ and $1 < \alpha$ are those induced by rotations.

RÉSUMÉ. Nous montrons que les seuls opérateurs de composition isométriques sur les espaces de type Bloch B^α pour $0 < \alpha < 1$ et $1 < \alpha$ sont ceux induits par rotation.

1. **Introduction and Preliminaries.** Let $\mathbb{D}$ be the unit disk in the complex plane and let $H(\mathbb{D})$ denote the space of functions analytic on $\mathbb{D}$. For an analytic function ϕ that maps the unit disk $\mathbb{D}$ into itself and for f in $H(\mathbb{D})$ define the *composition operator* C_ϕ by $C_\phi f = f \circ \phi$. We will say that the composition operator C_ϕ is nontrivial whenever ϕ is not a constant function. Composition operators can act on various types of function spaces. In each case the main goal is to discover the connection between the properties of the inducing function ϕ and the operator theoretic properties of C_ϕ. Extensive references for many of the known results on the subject can be found in [4] and [12].

In this paper we determine the functions ϕ that induce isometric composition operators on the Bloch-type spaces.

For $\alpha > 0$, the α-*Bloch spaces* B^α (or Bloch-type spaces) are the spaces of functions f analytic on $\mathbb{D}$ and such that

$$|||f|||_{B^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

Each B^α is a Banach space with norm of f given by $\|f\|_{B^\alpha} = |f(0)| + |||f|||_{B^\alpha}$. Note that for $\alpha = 1$, $B^1 = B$ is the classical Bloch space. For more references and details on the next few general facts about Bloch-type spaces stated below, see [14], [15].

Since for f in $H(\mathbb{D})$ we have that $f(z) = z\int_0^1 f'(zt)\,dt + f(0)$, the growth condition of a function f in B^α with $f(0) = 0$ is determined by the inequality $|f(z)| \le \|f\|_{B^\alpha} \int_0^1 \frac{|z|dt}{(1-|z|t)^\alpha}$. Thus for $\alpha = 1$, we get that

$$|f(z)| \le \|f\|_B \log \frac{1}{1-|z|};$$

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for $\alpha > 1$ we have that

$$|f(z)| \leq \frac{1}{\alpha - 1} \|f\|_{B^\alpha} \frac{1}{(1-|z|)^{\alpha-1}},$$

and for $0 < \alpha < 1$ we conclude that the functions in B^α are bounded analytic functions. Moreover, the spaces B^α with $0 < \alpha < 1$ are Lipschitz spaces $\mathrm{Lip}_{1-\alpha}$ and so $B^\alpha \subset A(\mathbb{D}) \subset H^\infty$, where $A(\mathbb{D})$ is the disk algebra.

For $0 < \alpha < \beta$, we have that $B^\alpha \subset B^\beta$. Thus the α-Bloch spaces B^α form an increasing, uniform family of function spaces. Note also that the Bloch space B contains H^∞ and is included in all of the Bergman spaces L_a^p, $p \geq 1$, while for large α, such as $\alpha \geq 2$, B^α includes the Bergman space L_a^2.

The boundedness (and compactness) of the composition operators on B^α has been established by J. Xiao [13]. The boundedness of composition operators on B^α for $0 < \alpha < 1$ was originally determined by K. M. Madigan [8].

Theorem A ([8], [13]). *For $\alpha > 0$ and ϕ an analytic self-map of $\mathbb{D}$, the composition operator C_ϕ is bounded on B^α if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |\phi'(z)|}{(1-|\phi(z)|^2)^\alpha} < \infty.$$

Note that for $\alpha \geq 1$, it follows from the Schwartz–Pick lemma that every composition operator is bounded on B^α. Since all of the spaces B^α include the identity function, a necessary condition for C_ϕ to be bounded is that ϕ belongs to B^α. Thus, for $0 < \alpha < 1$ every analytic self-map of $\mathbb{D}$ that is in $H^\infty \setminus B^\alpha$ induces an unbounded composition operator on B^α.

As we mentioned earlier, in this paper we will determine the isometric composition operators on B^α, *i.e.,* the operators C_ϕ such that

$$\|C_\phi f\|_{B^\alpha} = \|f\|_{B^\alpha}, \forall f \in B^\alpha.$$

Example. Let $\phi(z) = \lambda z, \forall z \in \mathbb{D}$, with $|\lambda| = 1$, *i.e.,* let ϕ be a rotation. Then C_ϕ is an isometry on all Bloch-type spaces, since for every $\alpha > 0$

$$\|C_\phi f\|_{B^\alpha} = \|f \circ \phi\|_{B^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(\lambda z)||\lambda|(1-|z|^2)^\alpha = \|f\|_{B^\alpha}.$$

The classification of isometric composition operators on the Bloch space, namely the case $\alpha = 1$, has been determined by F. Collona (see [3]).

Theorem B. [3] *Let ϕ be an analytic self-map of $\mathbb{D}$. The composition operator C_ϕ is an isometry on the Bloch space B if and only if $\phi(0) = 0$ and $\sup_{z\in\mathbb{D}} |\phi'(z)|(1-|z|^2) = 1$.*

Our main result takes care of all of the other Bloch-type spaces.

THEOREM C. *Let $0 < \alpha < 1$, $\alpha \neq 1$ and let ϕ be an analytic self-map of $\mathbb{D}$. Then the composition operator C_ϕ is an isometry on B^α if and only if ϕ is a rotation.*

Before we proceed with the proof of our result, let us say a bit more about general isometric operators and about isometric composition operators on some other spaces.

Composition operators are closely connected to surjective isometries and some general isometries on many spaces of functions. One of the oldest results is Banach's [1] characterization from 1932 of surjective isometries on the space of continuous functions over compact metric spaces as weighted composition operators. Similar results were derived by Forelli [6] in 1964 for the Hardy H^p spaces, by Kolaski [7] in 1981 for the Bergman L^p_a spaces, and by Cima and Wogen [2] in 1980 for the Bloch and little Bloch spaces. There are some results on general isometries that are not necessarily surjective, but the problem is usually much harder if there are no other restrictions on the operator, or if the space is not a Hilbert space. For more results and references on isometries on Banach spaces see [5].

As for the results on isometric composition operators, the surjectivity restriction oversimplifies the classification, since the nontrivial composition operators are always injective. Thus, one is mostly interested in determining the general isometric composition operators. Since on most spaces rotations induce isometric composition operators, it is of particular interest to determine the function spaces on which these are the only kind of isometric composition operators. Such are, for example, all of the weighted Bergman spaces, as shown by M. Martin and D. Vukotić [10].

One of the earliest results on isometric composition operators is Nordgren's result[11] from 1968 stating that if ϕ is inner, then C_ϕ is an isometry on H^2 if and only if $\phi(0) = 0$. Martin and Vukotic [10] have generalized recently that, indeed, C_ϕ is an isometry on $H^p, p \geq 1$, if and only if ϕ is inner and $\phi(0) = 0$. They have also classified the isometric composition operators on the Dirichlet space and, under the univalence condition of the inducing function, on some of the other Besov spaces (see [9]). Colonna's result [3], as stated above, shows that there are a variety of isometric composition operators on the Bloch space.

2. Proof of the Main Result.

We will show that the only isometric composition operators on the Bloch-type spaces, other than the Bloch space, are induced by rotations. In the proof, we use two different ideas for the cases $0 < \alpha < 1$ and $\alpha > 1$, and we divide the theorem correspondingly. First we prove a lemma.

LEMMA 2.1. *If C_ϕ is an isometry on B^α with $\alpha > 0$, then $\phi(0) = 0$.*

PROOF. The identity function $e(z) = z, \forall z \in \mathbb{D}$, belongs to each Bloch-type space, and $\|e\|_{B^\alpha} = 1$. Thus, since C_ϕ is an isometry, $\|C_\phi e\|_{B^\alpha} = \|\phi\|_{B^\alpha} = 1$.

Suppose that $\phi(0) = a \neq 0$. Using the function $f_a(z) = 1 - \bar{a}z$, we see that

$$\begin{aligned}\|C_\phi f_a\|_{B^\alpha} &= \|f_a \circ \phi\|_{B^\alpha} = |f_a(a)| + \sup_{z\in\mathbb{D}} |f_a'(\phi(z))|\,|\phi'(z)|(1-|z|^2)^\alpha \\ &= 1 - |a|^2 + |a| \sup_{z\in\mathbb{D}} |\phi'(z)|(1-|z|^2)^\alpha = 1 - |a|^2 + |a|(1-|a|) \\ &= 1 + |a| - 2|a|^2.\end{aligned}$$

But since C_ϕ is an isometry and $\|f_a\|_{B^\alpha} = 1 + |a| \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha = 1 + |a|$, it must be that $-2|a|^2 = 0$ and we get a contradiction to $a \neq 0$. Thus, $\phi(0) = 0$. □

Case $0 < \alpha < 1$. For the proof of the characterization of isometric composition operators on spaces B^α with $0 < \alpha < 1$ we use the fact that $B^\alpha = \mathrm{Lip}_{1-\alpha}$ and that their norms are equivalent. We also use the n-th iteration of ϕ defined by $\phi^{(n)} = \phi \circ \phi \circ \cdots \circ \phi$, n times.

Theorem 2.2. *Let $0 < \alpha < 1$ and let ϕ be an analytic self-map of $\mathbb{D}$. Then the composition operator C_ϕ is an isometry on B^α if and only if ϕ is a rotation.*

Proof. As already shown above, if ϕ is a rotation, then C_ϕ is an isometry, and so we only have to prove the other implication.

Let C_ϕ be an isometry on B^α. By Lemma 2.1, we have that $\phi(0) = 0$ and, as noted in the proof of the lemma, $\|\phi\|_{B^\alpha} = 1$. For every n in $\mathbb{N}$ and every $\phi^{(n)}$ as defined above, we have that $C_{\phi^{(n)}}$ must also be an isometry, and so $\|\phi^{(n)}\|_{B^\alpha} = 1$. Since B^α and $\mathrm{Lip}_{1-\alpha}$ have equivalent norms, we get that for every n

$$|\phi^{(n)}(z) - \phi^{(n)}(w)| \leq c\|\phi^{(n)}\|_{B^\alpha}|z-w|^{1-\alpha} \leq c|z-w|^{1-\alpha}, \quad \forall z, w \in \overline{\mathbb{D}}.$$

Thus, the sequence $\{\phi^{(n)}\}$ is an equicontinuous family in $C(\overline{\mathbb{D}})$. Since also $\|\phi^{(n)}\|_\infty \leq 1$, $\{\phi^{(n)}\}$ is also uniformly bounded in $C(\overline{\mathbb{D}})$. Using the Arzela–Ascolli theorem, let $\{\phi^{(n_k)}\}$ be the (uniformly on $\overline{\mathbb{D}}$) convergent subsequence.

Suppose now that ϕ is not a rotation. We will get a contradiction to the assumption that C_ϕ is an isometry by using the dynamical behavior of $\{\phi^{(n)}\}$ and the compactness criteria for composition operators on B^α. Since $\phi(0) = 0$, we must have that $\{\phi^{(n_k)}\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, as shown, for example, in [12, Proposition 1, p. 79]. Since $\{\phi^{(n_k)}\}$ converges uniformly on $\overline{\mathbb{D}}$, it must be that it converges to the zero function. But then for some k_0 we have that $\|\phi^{(k_o)}\|_\infty \leq r_0 < 1$, which implies that $C_{\phi^{(k_0)}}$ is compact on B^α (see [13]). Now a nontrivial composition operator is always injective and thus has a closed range whenever it is bounded below, which is the case when it is an isometry. On the other hand, a compact composition operator has a closed range only if it has a finite rank, which cannot happen for a nontrivial composition operator. But if $\phi^{(k_0)} = 0$, then $C_\phi^{(k_0)}$ cannot be an isometry. Hence ϕ must be a rotation. □

CASE $\alpha > 1$. The proof of the characterization of isometric composition operators on spaces B^α with $\alpha > 1$ relies on the Schwartz–Pick lemma, which states that if ϕ is a self-map of $\mathbb{D}$, then

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|\phi'(z)|}{(1-|\phi(z)|^2)} \leq 1,$$

and there is equality at one point if and only if there is equality at every point in $\mathbb{D}$ if and only if ϕ is a disk automorphism. See, for example, [12, p. 59] for more details.

THEOREM 2.3. *Let $\alpha > 1$ and let ϕ be an analytic self-map of $\mathbb{D}$. Then the composition operator C_ϕ is an isometry on B^α if and only if ϕ is a rotation.*

PROOF. As mentioned before, we only have to prove that if C_ϕ is an isometry, then ϕ must be a rotation. By Lemma 2.1, we have that $\phi(0) = 0$ and that $\|\phi\|_{B^\alpha} = 1$. Thus

$$1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\phi'(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1}\frac{(1-|z|^2)|\phi'(z)|}{(1-|\phi(z)|^2)}(1-|\phi(z)|^2).$$

Since $\alpha > 1$, and by the Schwartz–Pick lemma, we have that all three factors in the last product are smaller or equal to 1. They are also all continuous functions on $\mathbb{D}$, with $(1-|z|^2)^{\alpha-1}$ converging to 0, as $|z|$ approaches 1. Hence, the supremum must be attained at some point in $\mathbb{D}$. However, again by the Schwartz–Pick lemma, ϕ must be a disk automorphism with $\phi(0) = 0$, and so ϕ must be a rotation. □

To conclude, let us also mention that the family of Bloch-type spaces exhibits a peculiar behavior with regard to the isometric composition operators. There are other examples of problems related to composition operators where the behavior of the operator changes for a particular range of spaces in the family, usually either as the spaces become smaller, or as they become larger. Here, the only difference occurs for the Bloch space. As we can see in [3], isometric composition operators on the Bloch space are either induced by rotations, or by self-maps of $\mathbb{D}$ with infinite sequence of zeros $\{z_n\}$ containing 0 and such that

$$\lim_{n\to\infty}\sup(1-|z_n|^2)|\phi'(z_n)| = 1.$$

But the second kind of functions cannot belong to the Bloch-type spaces $B_\alpha, 0 < \alpha < 1$, because they are not continuous on $\overline{\mathbb{D}}$. So one can reason that the fact that the rotations are the only kind of functions inducing isometric composition operators on these types of spaces might not be surprising. For $\alpha > 1$ on the other hand, the spaces B^α are larger than the Bloch space and with (possibly) smaller norm of ϕ, which might be the reason why we do not have a similar variety of isometric composition operators as on the Bloch space.

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